Perturbative Approaches to 24

24 的微扰方法

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Abstract

摘要

We discuss the birth of the nonperturbative approach to quantum gravity known as quantum Einstein gravity, in which the gravitational interactions are conjectured to be asymptotically safe. The interactions are assumed to be finite and consistent at high energies, thanks to a scale-invariant ultraviolet completion. We present the framework on the basis of perturbative arguments that originally motivated it, paying special attention to the ε -expansion in $d = 2 + \varepsilon$ dimensions and the large- N expansion for N the number of flavors of matter fields. The chapter is organized in such a way that each section is mostly independent and can offer several ideas for both conceptual and technical future developments.

本文探讨被称为量子爱因斯坦引力的非微扰量子引力方法的起源，该理论认为引力相互作用是渐近安全的。依靠标度不变的紫外完备化，该理论假定相互作用在高能量下是有限且自洽的。我们基于最初推动该框架建立的微扰论证介绍了该框架，尤其关注 ε 展开 (在 $d = 2 + \varepsilon$ 维下) 以及物质场味数 N 的大 N 展开。本章的结构特点是各节大多相互独立，可为未来概念与技术层面的发展提供诸多思路。

Keywords

关键词

Quantum gravity - Asymptotic safety - UV completion - Perturbation theory . Analytic continuation

量子引力 - 渐近安全 - 紫外完备性 - 微扰论。解析延拓

Introduction

引言

The low energy description of the gravitational interactions at human, astrophysical, and cosmological scales has been successfully encoded in the classical field theory of general relativity, which adopts the metric field as the main ingredient with a redundant gauge formulation based on the group of diffeomorphisms. There is, in principle, the need to reconcile this description with our understanding of the quantum world, which contains quantum matter, and, eventually, obtain a quantum theory of the gravitational interactions. The simplest idea in this direction would be to promote the metric field and the Einstein-Hilbert action to a path integral and check if this lift makes sense. Here we are under the assumption that any meaningful canonical approach to gravity's quantization, for example, in terms of a wave functional of a metric, could be constructed only provided an ultraviolet complete description is available, or else it would not work at arbitrarily high energies.

人类尺度、天体物理尺度和宇宙学尺度引力相互作用的低能描述已经成功地编码在广义相对论经典场论中；该理论以度规场为核心组成部分，采用基于微分同胚群的冗余规范表述。原则上，我们需要将这一描述与我们对包含量子物质的量子世界的认知统一起来，并最终得到引力相互作用的量子理论。沿这个方向最朴素的思路是将度规场和爱因斯坦-希尔伯特作用量提升为路径积分，检验这一提升是否自洽。在此我们假设: 对引力量子化的任何合理正则方案 (例如以度规波泛函表述的方案)，只有存在紫外完备描述时才能构建，否则它无法在任意高能量下成立。

It has been well-known since a long time ago that (the path integral of) Einstein-Hilbert gravity is not perturbatively renormalizable, because it leads to on-shell nonrenormalizable divergences that appear at two loops for pure gravity [42,43,85], and at one loop for gravity with a cosmological constant [17] or gravity coupled to matter [9, 27] and gauge fields [28]. From a field-theoretical point of view, the implication is that the weakly coupled Gaussian limit of Einstein's metric gravity cannot be the ultraviolet limit of a theory of quantum gravity, in stark contrast to what happens to asymptotically free theories such as Yang-Mills gauge theories for certain gauge groups and number of flavor fields. Historically, this understanding has spawned a great deal of work in alternative approaches in which either the fundamental degrees of freedom were changed (e.g., string theory) or the methods were revisited (e.g., loop quantum gravity). The early days of metric quantum gravity are covered in section "The Failure of Perturbative Renormalizability" of this chapter, starting from the seminal work of 't Hooft and Veltman [82].

早在很久以前人们就已经知道，爱因斯坦-希尔伯特引力 (的路径积分) 不是微扰可重整化的: 这是因为它会产生在壳不可重整发散，纯引力的发散出现在两圈阶 [42,43,85]，带宇宙常数的引力 [17] 或与物质耦合的引力 [9, 27] 与规范场耦合的引力 [28] 的发散出现在一圈阶。从场论的角度来看，这意味着爱因斯坦度规引力的弱耦合高斯极限不可能是量子引力理论的紫外极限，这与杨-米尔斯规范理论这类渐近自由理论形成鲜明对比——对于特定规范群和味场数的杨-米尔斯理论而言，弱耦合高斯极限就是其紫外极限。在历史上，这一认知催生了大量替代方向的研究工作，这些工作要么改变了基本自由度 (例如弦论)，要么重新审视了量子化方法 (例如圈量子引力)。本章从特胡夫特和韦尔特曼的开创性工作 [82] 开始，在“微扰可重整化性的失败”一节介绍了度规量子引力的早期发展。

A less-known fact is that the construction of a low-energy effective quantum theory of gravity by standard path-integral methods is still meaningful if the loop expansion in the (reduced) Planck’s constant $\hbar$ is replaced by an effective expansion in some mass scale, which is generally identified with Planck’ s mass M_{Pl} [13, 29] . In this case, the ”effects” of the previously unwanted divergences of perturbation theory are suppressed by powers of E/M_{Pl} , where E is some typical energy scale of any process under consideration. This procedure leads to a well-defined effective field theory of quantum gravity whose effects can be computed systematically to any desired order in the expansion in powers of E/M_{Pl} and can be applied in the limit $E \ll M_{\mathrm{Pl}}$. Given the broad applicability of the methods of effective field theory, it stands natural that any ultraviolet consistent quantum theory of gravity should be expected to reproduce the effective field theory in the infrared limit.

一个较少为人知的事实是: 如果将 (约化) 普朗克常数 $\hbar$ 下的圈展开替换为某质量尺度 (通常取为普朗克质量 M_{Pl} [13, 29]) 的有效展开，通过标准路径积分方法构建引力的低能有效量子理论依然是有意义的。在这种情况下，微扰论中原本不受欢迎的发散，其“效应”会被 E/M_{Pl} 的幂次压低，其中 E 是所研究过程中某个典型能量标度。这一流程可以得到定义良好的量子引力有效场论，该理论的效应可以按 E/M_{Pl} 幂次展开系统计算到任意期望阶数，并且适用于 $E \ll M_{\mathrm{Pl}}$ 极限。鉴于有效场论方法的广泛适用性，任何紫外自洽的量子引力理论自然都应当在红外极限下退化为该有效场论。

In a rather straightforward way, the simplest reason why Einstein-Hilbert gravity is not perturbatively renormalizable is that the coupling - Newton’ s constant G_N - is dimensionful and has negative mass dimension, implying that the theory is not power-counting renormalizable in the standard sense. The observation that there are on-shell divergences that cannot be renormalized at one and two loops starting from an expansion of the Gaussian theory simply fits the original premise of the lack of perturbative renormalizability. The dimensionful nature of Newton’s constant also suggests implicitly the effective status of the metric’s quantum theory in the infrared.

很直接地看，爱因斯坦-希尔伯特引力不是微扰可重整化的最简单原因在于: 其耦合常数——牛顿常数 G_N ——是有量纲的，且质量维度为负，这说明该理论在标准意义上不满足幂计数可重整化。从高斯理论的展开出发，会出现在一、两圈阶无法重整化的在壳发散，这一结论正好契合引力不满足微扰可重整化性的最初推断。牛顿常数的有量纲性质也间接说明，度规量子理论在红外区本身就是一种有效理论。

A way around the lack of perturbative renormalizability is to study whether there are chances that a quantum theory of gravity might instead be nonperturbatively renormalizable. This idea has originally been framed by Weinberg [87], even before the perturbative nonrenormalizability was completely established. The idea, which now goes under the name of asymptotic safety conjecture, is funded on the premise that a metric theory of quantum gravity is asymptotically safe, rather than asymptotically free like certain gauge theories.

The properties of an asymptotically safe theory are best described in terms of the renormalization group: it is assumed that the renormalization group flow of the model under consideration admits a scale-invariant solution which has a finite number of relevant deformations in the ultraviolet. In practical terms, the fixed point is a zero of the couplings' beta functions, and the number of relevant directions can be established by studying the stability matrix of the flow. The scale-invariant fixed point plays the role of an ultraviolet completion of the quantum theory that does not, in principle, require additional degrees of freedom besides the metric and the preexisting matter and gauge fields coupled to it. Note that asymptotic safety asymptotic safety is not a stranger to gauge theories: some specific classes of gauge-Yukawa theories in four dimensions have been shown to be asymptotically safe, i.e., interacting in the UV limit [57] using perturbation theory in the Veneziano limit of large color and flavor numbers. We discuss Weinberg's original formulation of asymptotic safety in section "Weinberg's Asymptotic Safety".

> 解决微扰不可重整化问题的一种思路是，研究引力量子理论是否有可能通过非微扰方式实现可重整化。这一想法最初由 Weinberg 提出 [87]，甚至早于微扰不可重整化被完全证实。该观点如今被称为渐近安全猜想，其基础前提是: 量子引力的度规理论是渐近安全的，而非像部分规范理论那样是渐近自由的。渐近安全理论的性质可以通过重整化群得到最好的描述: 该猜想假设，所研究模型的重整化群流存在一个标度不变解，它在紫外区拥有有限个相关形变。实际来说，不动点是耦合 β 函数的零点，相关方向的数量可以通过研究流的稳定性矩阵确定。这个标度不变不动点承担了量子理论紫外完备化的作用，原则上，除度规以及与之耦合的原有物质场和规范场外，它不需要额外的自由度。需要注意的是，渐近安全对于规范理论并不陌生: 已经有研究证明，四维中某些特定类别的规范-汤川理论是渐近安全的，即在大颜色数和大味道数的 Veneziano 极限下，可通过微扰论证明其在紫外极限下存在相互作用 [57]。我们会在“温伯格的渐近安全”一节讨论温伯格对渐近安全的最初表述。

There are compelling reasons to believe the asymptotic safety conjecture, which are, somehow surprisingly, based on perturbative arguments. The first and most important for us is related to the fact that in two dimensions, Newton's constant is dimensionless, which hints at the fact that the quantum theory could be perturbatively renormalizable. In two dimensions the analysis is complicated by the fact that the Einstein-Hilbert action is a topological invariant and the only degree of freedom of the metric is the conformal one (the number of degrees of freedom changes discontinuously with the dimension), but these complications can be circumvented. The result is that the renormalization of Newton's constant gives a negative beta function, $\beta_{G_N} = -bG_N^2$ and $b > 0$, implying that the theory is asymptotically free. Now the crucial argument is that if we could continue the theory and its renormalization group flow above $d = 2$, we should replace $G_N \to \widetilde{G}_N \mu^{\frac{d-2}{2}}$, where μ is the scale of the running and $\widetilde{G}_N$ is Newton' s constant measured in units of μ . Under this assumption it is straightforward to see that the beta function of $\widetilde{G}_N$ has an ultraviolet fixed point $\widetilde{G}_N \sim \frac{\varepsilon}{b}$ where $d = 2 + \varepsilon$. The logic can be made almost flawless for $0 < \varepsilon \ll 1$ and, extrapolating it to $\varepsilon = 2$, leads naturally to the conclusion that quantum gravity should be asymptotically safe in $d = 4$ as argued by Weinberg himself [87]. This observation has spawned several important papers on two-dimensional gravity and the $d = 2 + \varepsilon$ expansion, including the results by Jack and Jones [53] and by Kawai and Ninomiya [49], which can be regarded as seminal works on their own. A partly historical account of the early results on $d = 2 + \varepsilon$ gravity is given in sections "Gravity in $d = 2 + \varepsilon$ " and "The Story of the Conformal Mode in $2 + \varepsilon$ Dimensions" of this chapter.

有令人信服的理由相信渐近安全猜想，某种程度上出人意料的是，这些理由都基于微扰论论证。对我们而言，第一条也是最重要的一条理由与二维下牛顿常数是无量纲量这一事实有关，这暗示量子引力理论可能是可微扰重整化的。在二维下，分析会因以下事实变得复杂: 爱因斯坦-希尔伯特作用量是拓扑不变量，度规唯一的自由度是共形自由度 (自由度的数量随维度不连续变化)，但这些复杂问题都可以规避。结果表明，牛顿常数的重整化给出负 beta 函数，$\beta_{G_N} = -bG_N^2$ and $b > 0$，意味着该理论是渐近自由的。现在关键的论点是，如果我们能将理论及其重整化群流延拓到 $d = 2$ 以上，我们就应当替换 $G_N \to \widetilde{G}_N \mu^{\frac{d-2}{2}}$，其中 μ 是跑动能标，$\widetilde{G}_N$ 是以 μ 为单位测量的牛顿常数。在这一假设下，我们可以很直接地看出，$\widetilde{G}_N$ 的 beta 函数存在一个紫外不动点 $\widetilde{G}_N \sim \frac{\varepsilon}{b}$，此时 $d = 2+\varepsilon$ 成立。这套逻辑对 $0 < \varepsilon \ll 1$ 几乎完全自洽，将其外推到 $\varepsilon = 2$ 后，就自然得出结论: 正如温伯格本人在文献 [87] 中所说，量子引力在 $d = 4$ 下应当是渐近安全的。这一观察催生了多篇关于二维引力和 $d = 2+\varepsilon$ 展开的重要论文，包括 Jack 与 Jones[53] 以及 Kawai 与 Ninomiya[49] 的研究结果，这些都可以视作开创性工作。本章“$d = 2+\varepsilon$ 维引力”和“共形模式在 $2+\varepsilon$ 维中的故事”两节对 $d = 2+\varepsilon$ 引力的早期研究结果给出了部分基于历史的综述。

A second compelling reason comes from the large- N analysis of Tomboulis [83] and Smolin [79], which shows that the gravitational fixed point of quantum gravity can be found in general dimension d - including four dimensions as special case - if one chooses to adopt an expansion in the large number of matter ”flavors” N (here the term matter includes all nongravitational fields) for which $\widetilde{G}_N \sim 1/N$ and further corrections could be computed as inverse powers of N . The large- N analysis gives the precious insight that the gravitational fixed point cannot be controlled only by the scalar curvature interaction through the Einstein-Hilbert action, but higher derivative operators that are nonzero on-shell (under the equations of motion of general relativity in the presence of matter) must be taken into account. In $d = 4$, the crucial operator to consider is the square of Weyl's curvature, whose interaction is renormalized by inverse powers of N , making the gravitational fixed point a higher derivative gravitational nonperturbative action in the large- N limit. The large- N expansion is summarized in section ”Large- N Expansion” following the seminal work of Smolin [79].

第二个引人注目的原因来自 Tomboulis[83] 和 Smolin[79] 的大 N 分析，该分析表明: 若采用大物质“味”数 N 展开 (此处物质包含所有非引力场)，量子引力的引力不动点可在任意维度 d (包括特殊情况四维) 中找到，在此框架下 $\widetilde{G}_N \sim 1/N$，进一步修正可按 N 的负幂次计算。大 N 分析给出了宝贵见解: 引力不动点无法仅通过爱因斯坦-希尔伯特作用量由标量曲率相互作用控制，必须考虑在 (含物质的广义相对论运动方程) 在壳上非零的高阶导数算符。在 $d = 4$ 中，需要考虑的关键算符是外尔曲率平方，其相互作用通过 N 的负幂次重整化，使得大 N 极限下引力不动点成为一个高阶导数引力非微扰作用量。大 N 展开遵循 Smolin[79] 的开创性工作，在“大 N 展开”一节中总结。

At this stage a clarification is in order. It is known since the work of Stelle [81] that higher derivative gravity is actually perturbatively renormalizable in $d = 4$. The higher derivative interactions that appear at quadratic order in the curvatures are, in fact, dimensionless, and it is possible to construct a sound loop-perturbative expansion using powers of their couplings, which appear as inverse powers multiplying the squared curvatures. In this case, the convergence of the expansion is improved by the fact that the propagators of the metric fluctuations scale as $1/p^4$ in momentum space, as opposed to $1/p^2$ of the standard actions with two derivatives. In perturbative higher derivative gravity, however, the Einstein-Hilbert term is just a relevant mass-like deformation of the corresponding Gaussian fixed point parametrized by Planck's mass square. This is in contrast to the fundamental role that the Einstein-Hilbert term has in the construction of the asymptotically safe fixed point; in fact the two solutions are believed to coexist [18,44]. Furthermore, the $1/p^4$ leads to potential problems with unitarity, which have been addressed in the past, but no generally accepted solution exists yet and the discussion is still open. Higher derivative gravity might then be asymptotically free and, possibly, unitary, at least if some conditions are met, so its perturbative expansion remains a captivating alternative to the solution provided by asymptotic safety. A brief report of some important results on higher derivative gravity and the modern approaches to circumvent its difficulties is given in section "Higher Derivative Gravity" of this chapter, but for a more detailed discussion of the renormalization, we refer to Ohta's contribution to this book [70].

在此需要澄清一点。自 Stelle[81] 的工作以来，人们已经知道在 $d = 4$ 中高阶导数引力实际是可微扰重整化的。出现在曲率二次阶的高阶导数相互作用实际上是无量纲的，可以利用其耦合常数的幂次构造合理的圈微扰展开，这些耦合常数表现为乘以曲率平方项的负幂次。这种情况下，展开的收敛性会因度量涨落的传播子在动量空间中标度为 $1/p^4$ 而得到改善，对比标准作用量中两阶导数对应 $1/p^2$ 的标度。但在微扰高阶导数引力中，爱因斯坦-希尔伯特项只是对应高斯不动点的一个相关类质量形变，由普朗克质量平方参数化。这与爱因斯坦-希尔伯特项在渐近安全不动点构造中的核心作用形成对比；目前认为这两种解可以共存 [18,44]。此外，$1/p^4$ 会带来潜在的幺正性问题，过去已有相关讨论，但尚未存在普遍接受的解决方案，讨论仍在开放进行中。因此高阶导数引力可以是渐近自由的，若满足一定条件甚至可能满足幺正性，所以它的微扰展开仍是渐近安全方案之外一个吸引人的替代选择。本章“高阶导数引力”一节简要介绍了高阶导数引力的一些重要结果，以及绕过其难点的现代研究方法，但关于重整化的更详细讨论，我们参考本书中 Ohta 的撰稿 [70]。

Coming back to asymptotic safety, it should be clear by now that the asymptotically safe fixed point is well-established both in the $d = 2 + \varepsilon$ and in the large- N expansions. Unfortunately both expansions do not apply straightforwardly at finite N and $\varepsilon = 2$, which are the physically interesting real-world limits, but rather require extrapolations. It could be possible, in principle, to compute several orders of the perturbative expansions and apply some resummation scheme, but this would require a considerable amount of work because the computation of diagrams with several loops, graviton lines, and derivative interactions is needed. Even setting aside the difficulty of the computation of subleading orders in the perturbative expansions, there is the additional deterrent that both the perturbative and the large- N expansions are most likely of asymptotic nature, which would also require a solution to the rather hard problem of reconstructing the full result from the perturbative series (as proposed, e.g., in the resurgent transseries program).

回到渐近安全，现在应该可以明确，渐近安全不动点在 $d = 2 + \varepsilon$ 展开和大 N 展开中都已经得到了很好的确立。遗憾的是，这两种展开都不能直接应用于物理上有意义的现实世界极限——有限值 N 和 $\varepsilon = 2$ ，而是需要外推。原则上，可以计算微扰展开的多阶结果并应用某种重求和方案，但这需要大量工作，因为需要计算含多圈、引力子线和导数相互作用的费曼图。即使不考虑微扰展开中次领头阶计算的难度，还有一个额外的障碍: 微扰展开和大 N 展开极有可能都是渐近级数，这还需要解决从微扰级数重构完整结果这个相当困难的问题 (例如，复兴跨级数方案中就提出了这一点)。

The way around to the limitations imposed by perturbation theory that has been adopted in the literature is to use a method for the computation of the renormalization group that is natively nonperturbative. The first nonperturbative computation of the renormalization group flow of Einstein-Hilbert gravity is due to Reuter [76], who used the functional renormalization group flow of an effective average action, which now goes under the name of Wetterich's equation [90], and the background field method in combination with covariant computational techniques based on the heat kernel. The application of the effective average action clearly shows an asymptotically safe fixed point under some approximations [80], which we are going to discuss momentarily. Ever since the result by Reuter, which now goes under the name of "quantum Einstein gravity," most of the asymptotic safety literature has been based on Wetterich's equation, improving over the original computation in several ways, which are too many to be accounted here, but a recent discussion on several aspects of the program can be found in [11]. This side of the literature is also covered at length by other authors in other chapters of this book.

文献中采用的绕开微扰论局限性的方法是使用一种本质为非微扰的重整化群计算方法。爱因斯坦-希尔伯特引力重整化群流的首次非微扰计算由 Reuter 完成 [76]，他使用了有效平均作用量的泛函重整化群流 (现称为 Wetterich 方程 [90])，并将背景场方法与基于热核的协变计算技术相结合。应用有效平均作用量后清晰表明，在一定近似下存在渐近安全不动点 [80]，我们很快就会对此展开讨论。自 Reuter 得到该结果 (如今被称为“量子爱因斯坦引力”) 以来，多数关于渐近安全的研究都基于 Wetterich 方程，从多个方面改进了最初的计算，这些改进数不胜数无法在此一一列举，关于该研究方向多个方面的最新讨论可参见文献 [11]。本书其他章节也由其他作者详细介绍了这部分研究内容。

The use of functional renormalization group method comes at a price. The main problem is that there is no perturbative coupling/parameter which can be used to estimate the size of any involved approximation. In the functional approach to renormalization, there is no ”bare” action, but rather one works directly with the effective average action that interpolates the effective action (the Legendre transform of the generator of connected diagrams) in the infrared; therefore, it is necessary to truncate the space of interactions of the average action for the computations to be practically doable. Ultraviolet and infrared modes are distinguished by means of a cutoff function, the form of which is, to some degree, arbitrary and potentially introduces a dependence on the computational scheme. Finally, the application to metric's gravity requires the use of the background field method which splits the metric, e.g., linearly as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, in a background $\bar{g}_{\mu\nu}$ over which fluctuations $h_{\mu\nu}$ are integrated: while the status of the average action as a function of an average field has been clarified in condensed matter applications in the past [89], it is not straightforward that the same property holds for average fluctuations $\langle h_{\mu\nu}\rangle$ over a background metric. In fact, results based on the effective average action tend to depend on gauge-fixing and other scheme-related parameters, although the evidence of the asymptotically safe fixed point is believed by most to be conclusive.

使用泛函重整化群方法是有代价的。核心问题在于，不存在可用于估计任何相关近似误差大小的微扰耦合/参数。在重整化的泛函方法中，不存在“裸”作用量，而是直接使用有效平均作用量开展工作，该作用量在红外内插有效作用量 (连通图生成泛函的勒让德变换)；因此，为了让计算实际可行，必须对平均作用量的相互作用空间做截断。紫外和红外模式通过截断函数区分，截断函数的形式在一定程度上是任意的，可能会引入对计算方案的依赖。最后，将该方法应用于度量引力需要使用背景场方法，该方法对度量做分解，例如线性分解为 $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ ，在背景 $\bar{g}_{\mu\nu}$ 上积分涨落 $h_{\mu\nu}$: 尽管过去在凝聚态应用中已经阐明了平均作用量作为平均场函数的性质 [89]，但背景度量上的平均涨落 $\langle h_{\mu\nu}\rangle$ 是否同样满足该性质并不明确。实际上，尽管多数研究者认为渐近安全不动点的证据确凿，但基于有效平均作用量得到的结果往往依赖于规范固定和其他与方案相关的参数。

For these reasons, at several points in time, some authors advocated for a ”parallel” renaissance of the perturbative approaches to metric gravity, to be discussed in tandem with the nonperturbative ones. The reason being that most of the ”problems” of the functional approaches can be discussed and solved relatively easily in the framework of perturbation theory, where it is straightforward to show that gauge and parametric dependencies cancel on-shell. Notably, this revival of perturbation theory included the works of Niedermaier [64-67], of Falls [30-32], and, more recently, of some of us using $d = 2 + \varepsilon$ [60, 61] . The application of perturbative methods highlights the importance of going on-shell when compiling the scaling properties of the effective action in the ultraviolet. It also shows that all unwanted parametric dependencies combine in the renormalization of the source term for the operator corresponding to the equations of motion, which, in turn, implies that the natural argument of the effective action is not simply the sum of the background metric and the expectation value of the fluctuations, $\langle g_{\mu\nu}\rangle \neq \bar{g}_{\mu\nu} + \langle h_{\mu\nu}\rangle$, but, rather, a more complicate functional operator; this is to be expected on the basis of the Vilkovisky-de Witt approach to the construction of an effective action that is covariant under reparametrizations of the fields. The perturbative approach also highlights potential caveats in the continuation of the dimensionality of spacetime d , both close to $d = 2$ and up and above $d = 4$. These have to do with the stability of the conformal mode, among other things, and might potentially induce new lines of research in the parallel approaches based on functional renormalization. For example, a general prediction of the revived perturbative approach is that there must be an upper critical dimension $d_{\rm cr}$ to the existence of the gravitational fixed point, which is not easy to see using the functional methods. The estimate from perturbation theory is that $d_{\rm cr} \approx 7.5$, but there can be sizeable corrections beyond the leading order in the expansion, so it is not straightforward to determine that $d_{\rm cr} > 4$, which is a necessary requirement for asymptotic safety in the physical case for obvious reasons. An account of the most recent works on $d = 2 + \varepsilon$ is given in section ”Revisiting Quantum Gravity in $2 + \varepsilon$ ”, while section ”Conclusions” offers an attempt to a conclusion.

基于这些原因，多个时期都有学者主张度量引力的微扰方法应当与非微扰方法并行讨论，迎来“平行”复兴。其原因在于，泛函方法的大多数“问题”都可以在微扰论框架下相对轻松地讨论和解决，在微扰论中可以直接证明规范依赖与参数依赖在在壳条件下抵消。值得注意的是，这场微扰论复兴包括 Niedermaier[64-67]、Falls[30-32] 的工作，以及更近阶段我们部分研究者利用 $d = 2 + \varepsilon$ [60, 61] 完成的工作。微扰方法的应用凸显了在整理有效作用量紫外标度性质时满足在壳条件的重要性。它还表明，所有不期望出现的参数依赖都整合进了对应运动方程算符源项的重整化中，这反过来意味着有效作用量的自然宗量不简单是背景度规加涨落期望值之和 $\langle g_{\mu\nu}\rangle \neq \bar{g}_{\mu\nu} + \langle h_{\mu\nu}\rangle$ ，而是一个更复杂的泛函算符；这一点在维尔科维斯基-德维特构造在场重参数化下协变的有效作用量的方案中本就是可预期的。微扰方法还指出了时空维数延拓 d 中潜在的问题，无论是接近 $d = 2$ 还是高于 $d = 4$ 的情况都是如此。这些问题尤其和共形模的稳定性相关，还可能为基于泛函重整化的平行研究方向带来新的研究思路。例如，复兴后的微扰方法有一个通用预言: 引力不动点存在上临界维数 $d_{\rm cr}$ ，这一点用泛函方法并不容易得到。微扰论给出的估计是 $d_{\rm cr} \approx 7.5$ ，但展开领头阶之外可能存在相当大的修正，因此无法直接确定 $d_{\rm cr} > 4$ ，而对真实物理情况的渐近安全来说，这一点显然是必要条件。关于 $d = 2 + \varepsilon$ 最新研究的介绍见“$2 + \varepsilon$ 中重探量子引力”章节，“结论”章节则给出了本文的总结。

The other sections of this contribution are meant to give a more in-depth discussion of the topics touched in this introduction as referenced above. Several of the computations that are presented in this chapter benefit from a better knowledge of the background field method and of the heat kernel, which are presented in sections ”Appendix A: Background Field Method” and ”Appendix B: Covariant Computations and the Heat Kernel”, respectively.

本文的其余部分将对引言中提及的这些主题进行更深入的讨论。本章介绍的多个计算都需要更深入了解背景场方法和热核，相关内容分别放在附录 A: 背景场方法和附录 B: 协变计算与热核两个章节中。

The Failure of Perturbative Renormalizability

微扰可重整性的破缺

The perturbative expansion of metric gravity starting from the bare Einstein-Hilbert action fails at one or two loops, depending on the presence of matter or the cosmological constant. The failure of perturbation theory is caused by divergences that cannot be reabsorbed in the renormalization of the original bare action, but in new operators that include higher powers of the curvatures.

> 从裸爱因斯坦-希尔伯特作用量出发的度规引力微扰展开，在 1 圈或 2 圈阶失效，具体取决于是否存在物质或宇宙学常数。微扰论失效源于无法被重新吸收到原始裸作用量重整化中的发散，这些发散只能出现在包含更高曲率幂次的新算符中。

In this section we review the first attempts of renormalizing general relativity. We start by reviewing the one-loop results following the seminal works of ’t Hooft and Veltman [82] for pure gravity and for gravity coupled to scalar fields, and of Christensen and Duff [17] for pure gravity with cosmological constant. Further historically relevant one-loop computations confirming and completing the results presented here can be found in [9, 27, 28] . We close this section by presenting the result found by Goroff and Sagnotti [42, 43] and van de Ven [85], i.e., the two-loop counterterm from which we learn that pure gravity is not perturbatively renormalizable in four dimensions. Some of the original works were performed in Lorentzian signature; however, for an easier comparison and a smoother exposition, we present all the results in their Euclidean signature counterpart.

> 本节我们回顾重整化广义相对论的早期尝试。我们首先回顾’t 霍夫特与韦尔特曼 [82] 针对纯引力、引力耦合标量场，以及克里斯坦森与达夫 [17] 针对带宇宙学常数的纯引力完成的开创性工作得到的 1 圈结果。更多证实并补全本文结果的具有历史意义的 1 圈计算可参考 [9, 27, 28] 。本节最后我们介绍戈罗夫与萨尼奥蒂 [42, 43] 以及范德文 [85] 得到的结果，即 2 圈抵消项，由此我们得知纯引力在四维下不具有微扰可重整性。部分原始工作是在洛伦兹号差下完成的，但为了便于比较、使表述更顺畅，我们所有结果都呈现为欧几里得号差下的对应形式。

The computation performed by ’t Hooft and Veltman is heavily based on a general formula for one-loop counterterm in flat space. Given a general Lagrangian for complex scalar fields φ_i on a d -dimensional flat space without boundary

> ’t 霍夫特与韦尔特曼的计算高度依赖于平坦空间中 1 圈抵消项的通用公式。给定复标量场 φ_i 在无边界的 d 维平坦空间中的一般拉格朗日量

$$\mathcal{L} = \varphi_i^* \partial_\mu \partial^\mu \varphi_i + 2\varphi_i^* N_{ij}^\mu \partial_\mu \varphi_j + \varphi_i^* M_{ij} \varphi_j, \tag{1}$$

one can obtain the one-loop counterterm:

> 可以得到 1 圈抵消项:

$$\Delta\mathcal{L} = \frac{1}{(4\pi)^2 \varepsilon} \operatorname{Tr}\left(X^2 + \frac{1}{6} Y_{\mu\nu} Y^{\mu\nu} \right), \tag{2}$$

where we compactly denoted

此处我们做了紧凑记号:

$$X = M - N^{\mu}N_{\mu} - \partial_{\mu}N^{\mu}$$

$$Y_{\mu\upsilon} = \partial_{\mu}N_{\upsilon} - \partial_{\upsilon}N_{\mu} + N_{\mu}N_{\upsilon} - N_{\upsilon}N_{\mu} \tag{3}$$

$$\varepsilon = d - 4$$

For a modern introduction to the necessary computational techniques to obtain (2), the reader is referred to sections ”Appendix A: Background Field Method” and

如需了解得到 (2) 式所需计算技术的现代介绍，读者可参考附录 A: 背景场方法与

”Appendix B: Covariant Computations and the Heat Kernel” and references therein. Using this result it is possible to simplify the computation of one-loop counterterms for matter fields coupled to gravity by drastically reducing the amount of graphs that have to be explicitly computed.

附录 B: 协变计算与热核以及其中引述的文献。利用该结果，可以大幅简化引力耦合物质场的 1 圈抵消项计算，显著减少需要显式计算的图的数量。

The generalization of the theory (1) to curved space reads:

理论 (1) 推广到弯曲空间的形式为:

$$\mathcal{L}_s = \sqrt{g}\left(-g^{\mu\upsilon}\partial_{\mu}\varphi^*\partial_{\upsilon}\varphi + 2\varphi^*N^{\mu}\partial_{\mu}\varphi + \varphi^*M\varphi\right). \tag{4}$$

One can approach the renormalization of such a theory by listing all the possible counterterms with the appropriate mass dimensions. Notice that when renormalizing a four-dimensional theory, counterterms due to gravity include operators that are quadratic in curvature invariants. However, the term $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ can be ignored since in $d = 4$, one has that the quantity:

我们可以通过列出所有具有合适质量维度的可能抵消项来处理该理论的重整化。注意，对四维理论做重整化时，引力带来的抵消项包含曲率不变量二次型的算符。但项 $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ 可以忽略，因为在 $d = 4$ 中，有如下关系:

$$\sqrt{g}E = \sqrt{g}\left(R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2\right) \tag{5}$$

is a total derivative and its integral is the topological Euler character. Hence, one can eliminate the squared Riemann tensor in favor of the Ricci tensor and the Ricci scalar. The coefficients of the counterterms that do not depend on the curvature can be fixed by direct comparison with the flat space case (2). Then, one can consider the special case of a conformally flat metric:

该量是全导数，其积分是拓扑欧拉示性数。因此我们可以消去黎曼张量平方项，代之以里奇张量和里奇标量。不依赖于曲率的抵消项系数可以通过直接和平坦空间的情况 (2) 对比来确定。接下来我们可以考虑共形平坦度规的特殊情况:

$$g_{\mu\nu} = \delta_{\mu\nu}F,\ F = 1 - f \tag{6}$$

and f is an arbitrary function of spacetime. In this setup, the theory can be rewritten in the form (1) with the substitution:

且 f 是任意时空函数。在该设定下，理论可以经过代换改写为 (1) 的形式:

$$M \to FM,\ N^\mu \to FN^\mu + \frac{1}{2}F^{-1}\partial^\mu F. \tag{7}$$

This fixes the counterterm Lagrangian up to a term of the form:

这就将抵消项拉格朗日量确定到只差如下形式的一项:

$$\Delta\mathcal{L}_a = \sqrt{g}a\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right). \tag{8}$$

The last coefficient a can be determined upon computing the one-loop self-energy for the gravitational fluctuations due to the scalar fields. In dimensional regularization only one Feynman diagram contributes to a . The final result for the counterterm Lagrangian for the theory (4) is

最后一个系数 a 可以通过计算标量场引起的引力涨落的 1 圈自能确定。在维数正则化中，只有一个费曼图对 a 有贡献。理论 (4) 的抵消项拉格朗日量的最终结果为

$$\Delta\mathcal{L}_s = \frac{\sqrt{g}}{8\pi^2\varepsilon}\,\mathrm{Tr}\left\{\frac{1}{12}Y_{\mu\upsilon}Y^{\mu\upsilon} + \frac{1}{2}\left(M - N_\mu N^\upsilon - \nabla_\mu N^\upsilon - \frac{1}{6}R\right)^2 + \frac{1}{60}\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right)\right\}. \tag{9}$$

One can use a similar approach to renormalize quantum fluctuations of gravity. We could consider the Lagrangian:

可以用类似的方法处理引力量子涨落的重整化。我们可以考虑如下拉格朗日量:

$$\mathcal{L}_g = \sqrt{g}\left(-R - \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\right) \tag{10}$$

and split the quantum fields in background plus fluctuations

并将量子场拆分为背景加涨落

$$g_{\mu\upsilon} = \bar{g}_{\mu\upsilon} + h_{\mu\upsilon},\ \varphi = \overline{\varphi} + \phi. \tag{11}$$

Due to the gauge symmetry of the gravitational content, it is necessary to include gauge fixing and ghost Lagrangians that depend on the background (we denote with $\overline{\nabla}$ the background covariant derivative). The gauge fixing is

由于引力内容具有规范对称性，必须引入依赖背景的规范固定和鬼拉格朗日量 (我们用 $\overline{\nabla}$ 表示背景协变导数)。规范固定项为

$$\mathcal{L}_{gf} = -\frac{1}{2}\sqrt{\bar{g}}\bar{g}^{\mu\upsilon}F_\mu F_\upsilon,\; F_\mu = \overline{\nabla}_\alpha h^\alpha_\mu - \frac{1}{2}\overline{\nabla}_\mu h^\alpha_\alpha - \phi\partial_\mu\overline{\varphi}, \tag{12}$$

and the corresponding ghost Lagrangian is

对应的鬼拉格朗日量为

$$\mathcal{L}_{gh} = \sqrt{\bar{g}}\bar{C}_\mu\left(\delta^\mu_\upsilon\overline{\nabla}^2 - \bar{g}^{\mu\alpha}\bar{R}_{\alpha\upsilon} - \partial^\mu\overline{\varphi}\partial_\upsilon\overline{\varphi}\right)C^\upsilon. \tag{13}$$

Although it requires cumbersome manipulations, the theory $\mathcal{L} = \mathcal{L}_g + \mathcal{L}_{gf} + \mathcal{L}_{gh}$, up to second order in the fluctuations, can be recast in the same form as (4) and the corresponding counterterms are found to be

尽管计算操作繁琐，但理论 $\mathcal{L} = \mathcal{L}_g + \mathcal{L}_{gf} + \mathcal{L}_{gh}$ 在涨落二阶近似下，可以改写为和 (4) 相同的形式，得到的对应抵消项为

$$\Delta\mathcal{L} = \frac{\sqrt{\bar{g}}}{8\pi^2\varepsilon}\left\{\frac{9}{720}\bar{R}^2 + \frac{43}{120}\bar{R}_{\mu\upsilon}\bar{R}^{\mu\upsilon} + \frac{1}{2}\left(\bar{g}^{\mu\upsilon}\partial_\mu\overline{\varphi}\partial_\upsilon\overline{\varphi}\right)^2\right.$$
$$\left. -\frac{1}{12}\bar{R}\bar{g}^{\mu\nu}\partial_\mu\overline{\varphi}\partial_\nu\overline{\varphi} + 2\left(\overline{\nabla}_\mu\overline{\nabla}^\mu\overline{\varphi}\right)^2\right\}. \tag{14}$$

From this result, one can infer the one-loop counterterm for pure gravity by sending $\overline{\varphi} \to 0$ and subtracting (9) at $M = N = 0$. As a result, we have:

从该结果出发，通过令 $\overline{\varphi} \to 0$ 并减去 $M = N = 0$ 处的 (9) 式，我们可以推导出纯引力的单圈抵消项，最终得到:

$$\Delta\mathcal{L}_{pg} = \frac{\sqrt{\bar{g}}}{8\pi^2\varepsilon}\left(\frac{1}{120}\bar{R}^2 + \frac{7}{20}\bar{R}_{\mu\upsilon}\bar{R}^{\mu\upsilon}\right). \tag{15}$$

A few remarks are in order. First, we notice how employing Einstein's equations in vacuum, the counterterm Lagrangian for pure gravity vanishes and the theory is one-loop finite. We could have guessed this without any actual computation since, for dimensional reasons, the only counterterm that could survive on-shell is given by the square of the Riemann tensor, but, as we said earlier, this can be eliminated, thanks to the topological density (5).

这里做几点说明。首先我们注意到，利用真空爱因斯坦方程，纯引力的抵消项拉格朗日量为零，理论在单圈水平是有限的。我们其实不需要实际计算就能得到这个结论: 从量纲分析来看，在壳条件下仅存的可能抵消项由黎曼张量的平方给出，但正如我们之前提到的，借助拓扑密度 (5)，这一项可以被消去。

However, if we now switch on the scalar fields, we can see that the background equations of motion for the gravity matter system reduce to

然而，如果我们引入标量场，就会发现引力-物质系统的背景运动方程可以约化为

$$\overline{\nabla}_\mu \overline{\nabla}^\mu \overline{\varphi} = 0,\ \bar{R}_{\mu\nu} = -\frac{1}{2}\overline{\nabla}_\mu \overline{\varphi} \overline{\nabla}_\nu \overline{\varphi}. \tag{16}$$

Inserting this into (14), we find that on-shell

将其代入 (14)，我们得到在壳条件下

$$\Delta\mathcal{L} = \frac{\sqrt{\bar{g}}}{8\pi^2\varepsilon}\frac{203}{80}\bar{R}^2, \tag{17}$$

hence, the gravity matter system is not one-loop renormalizable. This last point suggests that pure gravity is one-loop finite might be a coincidence due to the form of Euler character in four dimensions.

因此，引力-物质系统不是单圈可重整化的。这一点说明，纯引力在单圈水平有限可能只是四维空间中欧拉示性数的特殊形式带来的巧合。

It might be interesting to check how the renormalization of quantum gravity is affected by the inclusion of a cosmological constant. The theory is described by the action:

研究引入宇宙学常数后对量子引力重整化的影响是一个有意义的问题。该理论由如下作用量描述:

$$S_\Lambda[g] = -\frac{1}{16\pi G_N}\int \mathrm{d}^4x\sqrt{g}\,(R - 2\Lambda)\,, \tag{18}$$

and, at the classical level, is governed by the equations of motion

并且在经典水平由运动方程

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \tag{19}$$

Taking the trace of (19) one finds, in dimension $d = 4$, $R_{\mu\upsilon} = \Lambda g_{\mu\upsilon}$ and $R = 4\Lambda$. This gives the following simplification of the Euler density:

对 (19) 取迹可得，在维数 $d = 4, R_{\mu\upsilon} = \Lambda g_{\mu\upsilon}$ 和 $R = 4\Lambda$ 下，欧拉密度可以简化为:

$$E = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}. \tag{20}$$

Then, the one-loop computation is a straightforward generalization of the case without cosmological constant and can be performed using heat kernel methods in the background field method. The counterterm Lagrangian is given by

此时单圈计算就是无宇宙学常数情况的直接推广，可以在背景场方法下利用热核方法完成，得到的抵消项拉格朗日量为

$$\Delta\mathcal{L}_{\Lambda}=\frac{\sqrt{\bar{g}}}{180(4\pi)^{2}\varepsilon}\left(\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma}-2088\Lambda^{2}\right). \tag{21}$$

The term squared in the Riemann tensor is topological. One could therefore decide to include a term of the form $\alpha\int E$ in the original action without changing the equations of motion, and absorb the first term of (21) in the renormalization of α. The second term contributes to the running of the cosmological constant which will exhibit a quadratic beta function. The result (21) can also be found considering a generic field theory in curved space and specializing the matter content to the degrees of freedom propagated by general relativity [35]. Note that, in four dimensions, a constant redefinition of the background metric of the form $\bar{g}_{\mu\nu}=\omega\bar{g}'_{\mu\nu}$ can be absorbed in a rescaling of the coupling constants such that

黎曼张量的平方项是拓扑项，因此我们可以在原始作用量中加入一项形如 $\alpha\int E$ 的项，它不改变运动方程，就可以将 (21) 的第一项吸收到 α 的重整化中。第二项会贡献宇宙常数的跑动，使其出现二次贝塔函数。结果 (21) 也可以通过研究弯曲空间中的一般场论，再将物质内容限定为广义相对论传播的自由度得到 [35]。注意，在四维空间中，形式为 $\bar{g}_{\mu\nu}=\omega\bar{g}'_{\mu\nu}$ 的背景度量常数重定义可以被吸收到耦合常数的重新标度中，满足

$$G_N\to\frac{G_N}{\omega},\ \Lambda\to\Lambda\omega \tag{22}$$

however, the combination $\gamma=G_N\Lambda$ is left invariant [45]. Choosing $\omega=G_N$, one can read the flow of γ directly from the counterterm of the cosmological constant in (21) (see [35]):

但组合 $\gamma=G_N\Lambda$ 保持不变 [45]。取 $\omega=G_N$ 后，我们可以直接从 (21) 中宇宙学常数的抵消项读出 γ 的流 (见 [35]):

$$\beta_\gamma=-\frac{58}{10}\gamma^2. \tag{23}$$

Again, pure gravity in four dimensions shows no formal issue at one-loop level, thanks to the form of the topological Euler density. It is crucial, however, to check how the theory behaves beyond the one-loop approximation. An explicit calculation shows that, at two loops, the on-shell divergence for the Einstein-Hilbert theory is [42, 43]:

四维纯引力再次在单圈水平没有出现形式上的问题，这同样得益于拓扑欧拉密度的特殊形式。但研究理论在单圈近似之外的行为至关重要。已有明确计算表明，爱因斯坦-希尔伯特理论在两圈水平的在壳发散为 [42, 43]:

$$\Gamma_\infty=\frac{1}{(4\pi)^4\varepsilon}\frac{209}{2880}\int d^4\sqrt{\bar{g}}\bar{R}^{\alpha\beta}_{\ \ \gamma\delta}\bar{R}^{\gamma\delta}_{\ \ \zeta\xi}\bar{R}^{\zeta\xi}_{\ \ \alpha\beta}. \tag{24}$$

It is now clear that pure gravity in dimension $d=4$ is not perturbatively renormalizable.

现在可以明确，$d=4$ 维的纯引力不是微扰可重整化的。

Higher Derivative Gravity

高阶导数引力

One possible solution to the problem outlined in section ”The Failure of Perturbative Renormalizability” is to reformulate the perturbative expansion in terms of a higher derivative action, which is quadratic in the curvatures, instead of the Einstein-Hilbert action, which would then be a relevant deformation of the higher derivative action.

在“微扰可重整性的破缺”一节中提出的问题，一种可能解决方案是改用曲率二次型的高阶导数作用量来重新表述微扰展开，爱因斯坦-希尔伯特作用量则成为该高阶导数作用量的相关形变。

The advantage of this choice is that the bare propagator has better convergence properties due to the higher powers of momenta flowing in the loops. However, the unitarity of the theory is not guaranteed any more and requires special considerations. In this section we review some of the main aspects of this approach, making contact with the problem of unitarity. The fundamental references are the original work from Stelle [81], Refs. [6, 34, 35] where some errors in the computation of the renormalization group of earlier publications are corrected, but also the more recent Refs. [22,23]. Higher derivative gravity can be asymptotically free or trivial depending on one coupling; the phase in which it is not asymptotically free is argued to be the physical one and investigated in [77, 78] aiming at the ultraviolet completion (Neglecting the problem of unitarity, because shut up and compute!). The Lee-Wick approach to the problem of unitarity is explained in [4], while the discussion of higher derivative gravity as a PT-symmetric theory appears in [58,59].

这一选择的优势在于，由于圈动量的幂次更高，裸传播子具有更好的收敛性质。但该理论的幺正性不再得到保证，需要专门讨论。本节我们回顾这一方法的若干核心方面，并结合幺正性问题展开讨论。基础参考文献包括 Stelle 的原创工作 [81]，纠正了早期文献中重整化群计算错误的文献 [6, 34, 35]，以及较新的文献 [22,23]。高阶导数引力可根据耦合常数的不同表现为渐近自由或平凡；已有研究论证非渐近自由的相才是物理相，[77, 78] 中针对紫外完备化对该相开展了研究 (忽略幺正性问题，闭嘴计算就完事了)。Lee-Wick 解决幺正性问题的方案在 [4] 中给出，而将高阶导数引力作为 PT 对称理论的讨论可见 [58,59]。

There are two formulations of higher derivative gravity that are important for our purposes: the conformal version and the nonconformal one. The conformal action is

就我们的研究目标而言，高阶导数引力有两种重要表述: 共形表述与非共形表述。共形作用量为

$$S_{CG}[g] = -2\alpha \int \mathrm{d}^d x \sqrt{g} \left[R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right], \tag{25}$$

which is on-shell equivalent to the integrated square of the Weyl tensor. Conformal gravity is invariant under Weyl transformations of the metric; it emerges, for example, as the one-loop effective action of massless Dirac fermions that are minimally coupled to curved space through vielbeins and spin connection.

它在壳上等价于 Weyl 张量平方的积分。共形引力满足度规 Weyl 变换下的不变性；例如，它可以是通过标架与自旋联络最小耦合到弯曲空间的无质量狄拉克费米子的一圈有效作用量。

The nonconformal action, also known as Stelle's action, is

非共形作用量也称为 Stelle 作用量，形式为

$$S_{hd}[g] = -\int d^4x\sqrt{g}\left[\frac{1}{2\lambda}C^2 - \frac{1}{\rho}E + \frac{1}{\xi}R^2 + \tau\nabla^2 R - \frac{1}{G}(R-2\Lambda)\right], \tag{26}$$

where E is the Euler density defined in (5), $C^2 = C_{\mu\upsilon\rho\sigma}C^{\mu\upsilon\rho\sigma}$ is the square of the Weyl tensor and $G = 16\pi G_N$. The higher derivative operators control the perturbative series and would be enough to inspect for asymptotic freedom. The term linear in R is included as a relevant deformation and to allow for the classical field equations to be solved by the flat space metric. The equations are

其中 E 是 (5) 式定义的欧拉密度，$C^2 = C_{\mu\upsilon\rho\sigma}C^{\mu\upsilon\rho\sigma}$ 是 Weyl 张量的平方，$G = 16\pi G_N$ 。高阶导数算子控制微扰级数，足以用于检验渐近自由。包含 R 的线性项既是为了引入相关形变，也是为了让经典场方程可以存在平直空间度分解。场方程为

$$\frac{1}{G}\left(R^{\mu\upsilon} - \frac{1}{2}Rg^{\mu\upsilon} + g^{\mu\upsilon}\Lambda\right) + \frac{1}{\lambda}\left[2\left(\frac{1}{3} - \frac{\lambda}{\xi}\right)R\left(R^{\mu\upsilon} - \frac{1}{4}g^{\mu\upsilon}R\right) + \frac{1}{2}g^{\mu\upsilon}R_{\alpha\beta}R^{\alpha\beta}\right.$$

$$\left. -2R^{\mu\alpha\upsilon\beta}R_{\alpha\beta} + \left(\frac{1}{3} + 2\frac{\lambda}{\xi}\right)\nabla^\mu\nabla^\upsilon R - \nabla^2 R^{\mu\upsilon} + \left(\frac{1}{6} - 2\frac{\lambda}{\xi}\right)g^{\mu\upsilon}\nabla^2 R\right] = 0.$$

(27)

The covariant quantization of the theory in the background field approach requires an appropriate gauge fixing. From a kinematical point of view, one can choose the gauge-fixing condition $F_\mu[h]$ in the standard manner; however, from a dynamical point of view, the gauge-fixing action will have to contain higher-order derivatives to suppress the gauge modes appropriately. These extra derivatives play the role of a (background) bilinear form H in the gauge-fixing action so we can write a gauge-fixed path integral in the form:

背景场方法下该理论的协变量化需要合适的规范固定。从运动学角度，可以按标准方式选取规范固定条件 $F_\mu[h]$；但从动力学角度，规范固定作用量必须包含高阶导数来适当地压制规范模式。这些额外导数在规范固定作用量中扮演 (背景) 双线性型 H 的角色，因此我们可以将规范固定后的路径积分写为:

$$Z[g] = \det H^{-\frac{1}{2}}\int \mathcal{D}h\mathcal{D}\overline{\eta}\mathcal{D}\eta\exp\left\{-S_{hd}[\bar{g}+h] - \frac{1}{2}F_\mu H^{\mu\upsilon}F_\upsilon - \overline{\eta}_\mu H^{\mu\lambda}\Delta^{gh}_{\lambda\upsilon}\eta^\upsilon\right\},$$

(28)

where η and $\overline{\eta}$ are the Faddeev-Popov ghost and antighost, respectively, and Δ^{gh} is found in the usual way starting from the gauge condition $F[h]$. In higher derivative gravity, the conformal transformation of R^2 generates two scalar modes: one with the standard kinetic sign of a scalar field and one corresponding to the usual conformal factor with opposite sign in the kinetic term. The one-loop effective action is found to be

其中 η 和 $\overline{\eta}$ 分别是法捷耶夫-波波夫鬼场和反鬼场，Δ^{gh} 可以从规范条件 $F[h]$ 按常规方式得到。在高阶导数引力中，R^2 的共形变换会生成两个标量模式: 一个模式具有标量场的标准动能符号，另一个对应通常的共形因子，动能项符号相反。一圈有效作用量结果为

$$\Gamma_{hd} = \frac{1}{2}\,\mathrm{Tr}\log\Delta_h - \mathrm{Tr}\log\Delta^{gh} - \frac{1}{2}\,\mathrm{Tr}\log H$$

$$\Delta_h^{\mu\nu\alpha\beta} = \left.\frac{\delta^2 S_{gh}}{\delta g_{\mu\nu}\delta g_{\alpha\beta}}\right|_{\bar{g}}. \tag{29}$$

An explicit calculation with dimensional regularization in $d = 4-\varepsilon$ dimensions leads to a counterterm action that includes the same operators as above:

在 $d = 4-\varepsilon$ 维中使用维正规化进行显式计算，会得到包含上述所有算子的抵消项作用量:

$$\Gamma_{ct} = \frac{\mu^\varepsilon}{16\pi^2\varepsilon}\int \mathrm{d}^4x\sqrt{\bar{g}}\left[\beta_1\bar{C}^2 - \beta_2\bar{E} + \beta_3\bar{R}^2 + \beta_4\bar{R} + \beta_5\right], \tag{30}$$

with the coefficients

系数为

$$\beta_1 = \frac{133}{20},\ \beta_2 = \frac{196}{45},\ \beta_3 = 10\frac{\lambda^2}{\xi^2} - 5\frac{\lambda}{\xi} + \frac{5}{36},$$

$$\beta_4 = \frac{\lambda}{G}\left(\frac{\xi}{12\lambda} - \frac{13}{6} - 10\frac{\lambda}{\xi}\right),\ \beta_5 = \frac{1}{G^2}\left(\frac{5}{2}\lambda^2 + \frac{\xi^2}{72}\right) + \frac{\Lambda\lambda}{G}\left(\frac{56}{3} - \frac{2\xi}{9\lambda}\right).$$

(31)

Notice that the coupling τ drops out of the counterterm action as expected from a total derivative.

注意耦合 τ 作为全导数会按预期从抵消项作用量中消失。

An important question concerns the physical content of the renormalization flow generated by (30). We can look for a ”gauge-invariant” combination of the couplings (which have beta functions with universal coefficients independent on gauge-fixing parameters). For the Einstein-Hilbert sector, this is given by the combination $\gamma = G_N\Lambda$ with β -function:

一个重要问题关乎 (30) 式生成的重整化流的物理内容。我们可以寻找耦合常数的“规范不变”组合 (这些组合的 β 函数具有与规范固定参数无关的普适系数)。对于爱因斯坦-希尔伯特部分，该组合由 $\gamma = G_N\Lambda$ 给出，对应的 β 函数为:

$$\beta_\gamma = -2\Lambda G_N^2\beta_4 - \frac{1}{2}G_N^2\beta_5. \tag{32}$$

In the higher derivative sector, a natural choice is

在高阶导数部分，一个自然的选择是

$$\theta = \frac{\lambda}{\rho}\ \omega = -3\frac{\lambda}{\xi}. \tag{33}$$

In $d = 4$ one finds two fixed points, one UV stable and one UV unstable, both for negative of ω . The UV stable fixed point corresponds to asymptotic freedom of the theory. A negative solution for ω is often disregarded on the ground that it leads to an unstable static gravitational potential. Note that the sign of ω is opposite to the sign of ξ since λ is required to be greater than zero requiring positivity of the theory.

在 $d = 4$ 中可找到两个不动点，一个紫外稳定，一个紫外不稳定，二者均对应 ω 为负的情况。紫外稳定不动点对应理论的渐近自由。ω 的负解通常因会导致不稳定的静态引力势而被忽视。请注意，ω 的符号与 ξ 的符号相反，因为要求 λ 大于零以保证理论的正定性。

What is, then, the physical picture corresponding to $\omega > 0$? One could study this scenario in the context of ”agravity” [78], i.e., only including dimensionless operators in the theory truncation (therefore dropping the Einstein-Hilbert sector) and coupling the theory to a matter sector (again, truncated to the dimensionless operators). The UV limit of such theory can reach a conformal theory of gravity provided that ξ diverges and all the other couplings remain asymptotically free. This happens because the two degrees of freedom of the conformal mode decouple at high energy and the limit $\xi \to \infty$ is consistent only when the coupling to scalar fields, $\mathcal{L}_\phi = -\frac{f_{ab}}{2}\phi_a\phi_b R$, reaches the Weyl-invariant value of $f_{ab} = -\frac{1}{6}\delta_{ab}$. Conversely, the Weyl-invariant high energy theory is not consistent since the symmetry is anomalous and, at lower energies, the conformal mode couples to those operators that break scale invariance through their β -functions. As a result there is an anomalous generation of finite values of ξ along the RG flow. According to [78], it is possible to exclude (at least perturbatively) that the coupling ξ hits a Landau pole in the high energy limit; however, nonperturbative computations would be needed to discern whether it hits a UV interacting fixed point or if ξ still grows to infinity. In any case we must always recall that the renormalization group actually only flows from UV to IR, and not the converse, so it would be appropriate for future developments of [78] to formulate the approach from a ”top-down” perspective.

那么，对应于 $\omega > 0$ 的物理图像是什么？我们可以在“无引力 (agravity)”[78] 的框架下研究这一情形: 即在理论截断中仅包含无量纲算符 (因此舍去爱因斯坦-希尔伯特部分)，并将该理论耦合到物质部分 (同样，物质部分也截断为仅保留无量纲算符)。若 ξ 发散，而所有其他耦合保持渐近自由，这类理论的紫外极限可以成为共形引力理论。这是因为共形模式的两个自由度在高能下退耦，且只有当耦合到标量场的 $\mathcal{L}_\phi = -\frac{f_{ab}}{2}\phi_a\phi_b R$ 达到 $f_{ab} = -\frac{1}{6}\delta_{ab}$ 的外尔不变值时，极限 $\xi \to \infty$ 才自洽。反之，外尔不变的高能理论并不自洽，因为该对称性是反常的；在更低能标下，共形模式会耦合到那些通过其 β 函数破缺标度不变性的算符。结果就是，ξ 的有限值会在重整化群流中因反常产生。根据文献 [78] 的结论，可以 (至少微扰层面上) 排除 ξ 在高能极限碰到朗道极点的可能；但仍需要非微扰计算来判断它是否会抵达紫外相互作用不动点，或是 ξ 仍会增长至无穷。无论如何我们必须始终牢记，重整化群实际上只从紫外流向红外，而非反向，因此未来对文献 [78] 的拓展应当从“自上而下”的视角来构建这一方法。

What about unitarity? Perturbatively, unitarity appears to be lost due to the propagation of extra ghost fields associated with the higher derivative propagator (kinematical ghosts are negative norm states of the S-matrix). If one tries to perform only analytical operations on the propagator defined by (26), the only possibility to restore unitarity seems to come at the price of unbounded energies in the theory spectrum. This feature is common in higher derivative field theories; however, in the context of Lee-Wick theories, it is possible to deal with these issues by performing a nonanalytic continuation of the tree-level propagator. Such a procedure, which consists in deforming the integration domain of Feynman graphs in momentum space, turns the extra ghosts into fake degrees of freedom known as fakeons [4]. Fakeons appear as purely virtual particles in the vertex expansion of correlators, but they do not appear in the physical spectrum. In this way one gets rid of ghosts and obtains a unitary theory without spoiling the boundedness of the energy spectrum.

As consequence of this prescription to compute correlators in higher derivative theories, microcausality is lost; however, it is possible to compute a time scale that bounds such a violation and macrocausality seems to be preserved.

幺正性又如何呢？在微扰层面，由于更高导数传播子对应额外鬼场的传播 (运动学鬼是 S 矩阵的负范数态)，幺正性似乎会丧失。如果仅尝试对式 (26) 定义的传播子做解析操作，恢复幺正性的唯一方法似乎是牺牲理论谱中能量的有界性。这个特征在高阶导数场论中十分常见；但在李-维克理论的框架下，我们可以通过对树级传播子做非解析延拓来处理这些问题。该过程将费曼图在动量空间的积分域形变，把额外鬼转化为被称为赝粒子 (fakeon) 的赝自由度。赝粒子在关联函数顶点展开中表现为纯虚粒子，但不会出现在物理谱中。通过这种方式，我们可以去除鬼，得到幺正理论，同时不破坏能谱的有界性。作为这个计算高阶导数理论关联函数方案的结果，微观因果性会丧失；但我们可以计算出一个限定这种破坏的时间标度，宏观因果性似乎得以保留。

A slightly different take on the topic of unitarity relies on the realization that knowing the propagator of a theory is insufficient to construct the Hilbert space. A careful analysis of (26) reveals that the Hamiltonian is not Hermitian but rather PT-symmetric (The Hamiltonian is actually CPT-symmetric, but since charge conjugation is preserved independently, it is also a PT-symmetric theory.) [58,59]. The choice of the vacuum of the theory is then what discerns between a scenario with unbounded negative energies from one that apparently is non-unitary. However, the correct choice of internal product for the Hilbert space reveals that (26) is actually free of ghosts, in the sense that they do not contribute with negative norm to the spectrum of the theory. Unfortunately, a negative aspect of this analysis is that the limit from (26) to pure quadratic gravity is singular and one cannot get rid of the Einstein-Hilbert sector to compare to the situation presented, e.g., in conformal pure gravity.

关于幺正性问题还有一个略有不同的研究思路，其核心观点是: 仅知道理论的传播子不足以构造希尔伯特空间。对式 (26) 的细致分析表明，哈密顿量不是厄米的，而是 PT 对称的 (哈密顿量实际上是 CPT 对称的，但由于电荷共轭是独立保持的，该理论也是 PT 对称的)[58,59]。理论真空的选择区分了两种情形: 一种是存在无界负能，另一种是表观上非幺正。然而，对希尔伯特空间内积的正确选择表明，式 (26) 实际上不存在鬼，也就是说鬼不会对理论谱贡献负范数。遗憾的是，该分析有一个缺陷: 从式 (26) 到纯二次引力的极限是奇异的，我们无法舍弃爱因斯坦-希尔伯特 sector，来和例如共形纯引力中的情形做对比。

Large-N Expansion

大 N 展开

The limit of large number of components of field multiplets typically leads to strong simplifications in otherwise inaccessible computations. We recall here interesting results in such limit when multicomponent matter fields are coupled to gravity.

场多重态分量数取大极限时，通常会极大简化原本无法进行的计算。我们在此回顾多分量物质场与引力耦合时，该极限下的有趣结果。

This approach to a quantum theory of gravity with improved renormalization properties relies on coupling the geometry to matter degrees of freedom. Typically, the number of matter fields of the model appears parametrically in the renormalization group flow of the gravitational sector, and the limit of large number of flavor species can be accessed with standard perturbative methods. This approach was initiated by Tomboulis in [83], where the analytical properties of the gravitational propagator have been studied. In Ref. [79], which we follow in this section, Smolin investigated the existence of an interacting ultraviolet fixed point. Further results on the large- N limit of quantum gravity can be found in [19,73], in which alternative nonperturbative analysis have been carried out. The following analysis focuses on the four-dimensional case; however, a similar study can be carried out in two dimensions. Some comments about the effect of matter couplings to the renormalization of gravity in $d = 2 + \varepsilon$ appear at the end of this section and refer to part of the work done in [79]. Further examples can be found in [62], where the analysis focuses on scalar matter rather than fermionic degrees of freedom.

这种改进了重整化性质的量子引力研究方法，依赖于将几何与物质自由度耦合。通常，该模型的物质场数目会作为参数出现在引力 sector 的重整化群流中，我们可以用标准微扰方法处理大味种类数极限。该方法由 Tomboulis 在文献 [83] 中开创，他在其中研究了引力传播子的解析性质。本节我们沿用文献 [79] 的工作，Smolin 在该文献中研究了相互作用紫外固定点的存在性。量子引力大- N 极限的更多结果可参见文献 [19,73]，其中完成了替代性非微扰分析。下文分析聚焦四维情形；不过类似研究也可以在二维中开展。关于 $d = 2 + \varepsilon$ 中物质耦合对引力重整化的影响，本节末尾会给出一些说明，这些内容源自文献 [79] 的部分工作。更多例子可参见文献 [62]，该文献的分析聚焦标量物质而非费米子自由度。

We start by considering the action for N massless spinors coupled dynamically to the geometry (in Lorentzian signature):

我们首先考虑 N 个无质量旋量与几何动力学耦合 (洛伦兹号差) 的作用量:

$$S = -\int \mathrm{d}^4 x \sqrt{g} \left[\frac{1}{G} R - \sum_{i=1}^{N} \overline{\psi}_i \nabla \psi_i + \frac{1}{2} \Lambda \right], \tag{34}$$

where ψ_i is a Dirac spin- 1/2 field and R is the Ricci scalar for the metric $g_{\mu\nu}$. We can study the UV behavior of the theory introducing a cutoff K and inspecting the divergences in the large- N limit via an expansion in $\frac{1}{N}$ of the Feynman diagrams. It is useful to introduce rescaled couplings as

其中 ψ_i 是狄拉克自旋- 1/2 场，R 是度规 $g_{\mu\nu}$ 的里奇标量。我们可以引入截断 K 来研究理论的紫外行为，并通过对费曼图做 $\frac{1}{N}$ 展开来分析大- N 极限下的发散。引入重新标度的耦合是很有用的:

$$\frac{1}{G} = \frac{cNK^2}{2}, \; \Lambda = N\lambda K^4, \tag{35}$$

and rewrite (34) as

并将 (34) 重写为

$$S = -\int \mathrm{d}^4 x \sqrt{g} \left[\frac{1}{2} cNK^2 R - \sum_{i=1}^{N} \overline{\psi}_i \nabla \psi_i + \frac{1}{2} \lambda N K^4 \right]. \tag{36}$$

A convenient expansion is in terms of dimensionless fluctuations $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, such that all the leading order divergences will appear in the graviton propagator, rather than in the graviton-matter vertex.

The drawback of this choice is that the flat Minkowski metric is not a solution of the equations of motion due to the presence of a cosmological constant rescaled by K^4 . To ensure a consistent renormalization, one would need to consider tadpole diagrams as well to renormalize the classical equations of motion. However in [79] it was shown that in the limit of vanishing renormalized cosmological constant, at leading order in $\frac{1}{N}$, the tadpole diagrams due to fermionic loops cancel against tadpoles due to the insertion of the gravitational one-point function. We therefore simplify the analysis presented here and assume that we have performed this choice of scheme.

一种方便的展开是基于无量纲涨落 $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ ，此时所有领头阶发散都会出现在引力子传播子中，而非引力子-物质顶点。这种选择的缺点是，由于存在被 K^4 重新标度的宇宙学常数，平坦闵氏度规不是运动方程的解。为了保证重整化自洽，我们还需要考虑蝌蚪图来重整化经典运动方程。但文献 [79] 表明，在重整化宇宙学常数趋于零的极限下，领头阶 $\frac{1}{N}$ 处，费米圈产生的蝌蚪图会与引力单点插入产生的蝌蚪图抵消。因此我们简化此处的分析，假设我们已经完成了这套方案选择。

At this point the analysis can be carried out by computing the renormalized propagator and requiring that the limit $K \to \infty$ exists. Introducing the projector on the spin-2 states $P^{(2)}_{\mu\nu\alpha\beta}$, the bare propagator for the gravitational sector is

至此，我们可以通过计算重整化传播子并要求 $K \to \infty$ 极限存在来完成分析。给自旋 2 态投影算子 $P^{(2)}_{\mu\nu\alpha\beta}$ ，引力 sector 的裸传播子为

$$D_{\mu\nu\alpha\beta}\left(p^2\right) = \frac{P^{(2)}_{\mu\nu\alpha\beta}}{cNK^2\left(p^2 + i\varepsilon\right)} + \dots \tag{37}$$

where the dots stand for gauge-dependent parts propagating the lower spin modes and we will not consider them here.

其中省略号代表传播低自旋模式的规范相关部分，我们在此不做讨论。

The leading order divergences in the $\frac{1}{N}$ expansion are given by the insertion of fermionic loops, but no gravitational loop. The result for the one-loop renormalized propagator in four dimensions is then given in [79]

$\frac{1}{N}$ 展开中的领头阶发散来自费米圈插入，不包含引力圈。因此四维中单圈重整化传播子的结果可见文献 [79]

$$D^{1/N}_{\mu\nu\alpha\beta}\left(p^2\right) = \frac{P^{(2)}_{\mu\nu\alpha\beta}}{cNp^2K^2\left[1 - \left(cNp^2K^2\right)^{-1}\left(iNF^{(2)} + \lambda NK^4\right)\right]} + \dots \tag{38}$$

$F^{(2)}$ is the spin-2 part of the fermionic loop insertion in the Feynman diagrams needed to renormalize the graviton propagator and contains polynomial terms in the cutoff K up to quartic order and a logarithmic term $\sim p^4 \log\left(-p^2/K^2\right)$. Requiring that the K dependence in the denominator disappears as in (38) fixes the value of c up to a term that vanishes in the limit $K \to \infty$:

$F^{(2)}$ 是重整化引力子传播子所需费曼图中费米圈插入的自旋 2 部分，它包含截断 K 最高到四次阶的多项式项，以及一个对数项 $\sim p^4 \log\left(-p^2/K^2\right)$。要求分母中对 K 的依赖如 (38) 一般消去，就可以确定 c 的值，仅相差一个在 $K \to \infty$ 极限下趋于零的项:

$$\frac{1}{c} = 32\pi^2 \left(1 - \frac{M^2}{K^2}\right). \tag{39}$$

Similarly, λ is fixed by requiring that the K^4 term in $F^{(2)}$ is canceled by the (divergent) bare cosmological constant:

类似地，要求 $F^{(2)}$ 中的 K^4 项被 (发散的) 裸宇宙学常数抵消，即可固定 λ:

$$\lambda = \frac{9}{128\pi^2} + \frac{L^4}{K^4} 32\pi^2. \tag{40}$$

From Eqs. (39) and (40), we see that the system $\{c, \lambda\}$ approaches a fixed point:

由式 (39) 和 (40) 可知，系统 $\{c, \lambda\}$ 趋近于一个不动点:

$$\{c^*, \lambda^*\} = \left\{\frac{1}{32\pi^2}, \frac{9}{128\pi^2}\right\}. \tag{41}$$

The rate at which the fixed point is approached is given by the combination $\sqrt{N}M$, which at low energies plays the role of an effective Planck mass, and by L. The requirement of vanishing bare cosmological constant is $L = 0$.

趋近不动点的速率由组合 $\sqrt{N}M$ 和 L 给出，其中 $\sqrt{N}M$ 在低能下扮演有效普朗克质量的角色。对裸宇宙学常数为零的要求是 $L = 0$。

We are now left with the logarithmic divergence coming from $F^{(2)}$. In order to cancel this term, we need to include a higher derivative operator in the bare Lagrangian density:

现在我们得到了由 $F^{(2)}$ 产生的对数发散。为了抵消这一项，我们需要在裸拉格朗日密度中引入一个高阶导数算符:

$$\mathcal{L}_C = \frac{N}{\alpha\left(\frac{M}{K}\right)} C^2 \tag{42}$$

where C^2 is the square of the Weyl tensor and α is a new bare parameter. The renormalization of α is taken care by the counterterm [79,83]:

其中 C^2 是外尔张量的平方，α 是一个新的裸参数。α 的重整化由 counterterm[79,83] 处理:

$$\Delta\mathcal{L} = N\left[\frac{1}{480\pi^2} \log\left(\frac{K^2}{M^2}\right) + \frac{1}{\alpha} - \frac{353}{150}\right] C^2. \tag{43}$$

From the counterterm (43), one can see that the new coupling is asymptotically free and the full system of couplings has the fixed point:

从 counterterm(43) 可以看出，这个新耦合是渐近自由的，完整的耦合系统存在不动点:

$$\{c^*, \lambda^*, \alpha^*\} = \left\{\frac{1}{32\pi^2}, \frac{9}{128\pi^2}, 0\right\}. \tag{44}$$

The computation of the renormalized propagator for the graviton in the new truncation leads to

在新截断下计算引力子的重整化传播子可得

$$D^{1/N}_{\mu\nu\alpha\beta} = \frac{P^{(2)}_{\mu\nu\alpha\beta}}{Np^2\left(M^2 - \frac{1}{\alpha p^2} - \frac{1}{480\pi^2} p^2 \log\left(-\frac{p^2}{M^2}\right)\right)} + \dots \tag{45}$$

The result (45) coincides with the one found in dimensional regularization, where only the coupling α gets renormalized as shown in [83].

结果 (45) 与维度正则化中得到的结果一致，如文献 [83] 所示，维度正则化中只有耦合 α 被重整化。

Before moving on it is worth mentioning that a similar analysis can be performed in dimension $d = 2 + \varepsilon$, as shown in [79]. The analysis carries through in a similar fashion as shown here up to few modifications. Again, the inclusion of a higher derivative operator, proportional to $R^2_{\mu\nu}$ and parametrized by the coupling α , is needed to cancel logarithmic divergences. The sector of Einstein-Hilbert couplings reaches a fixed point that can be understood as the continuation of (44) and reads:

在继续讨论之前，值得一提的是，如文献 [79] 所示，可以在 $d = 2 + \varepsilon$ 维中进行类似的分析。该分析除少量修改外，流程与本文给出的分析类似。同样，为了抵消对数发散，需要引入一个与 $R^2_{\mu\nu}$ 成正比、由耦合 α 参数化的高阶导数算符。爱因斯坦-希尔伯特耦合部分达到的不动点可以理解为 (44) 的延拓，形式为:

$$\{c^*, \lambda^*\} = \left\{\frac{1}{36\pi}, \frac{9}{32\pi}\right\}. \tag{46}$$

Although, as we will see in the next section, there are many indications of a UV fixed point of order $O(\varepsilon)$ for Newton’s constant in $d = 2 + \varepsilon$, the fixed point found in the large- N expansion is of order $O(1)$ and, thus, seems to be unrelated to the fixed point we will study in the next sections.

尽管我们下一节会看到，有诸多迹象表明 $d = 2 + \varepsilon$ 中牛顿常数存在 $O(\varepsilon)$ 阶的紫外不动点，但大- N 展开中得到的这个不动点是 $O(1)$ 阶的，因此似乎与我们后续章节将要研究的不动点无关。

Weinberg’s Asymptotic Safety

温伯格渐近安全

We present the general ideas at the basis of Weinberg’s asymptotic safety paradigm which extends the scope of renormalizable quantum field theories. The paradigm draws heavily from the interpretation of the renormalization group that emerges from the theory of critical phenomena, which represents a significant departure from the purely pragmatic perspective that was applied to particle physics before Weinberg.

我们介绍温伯格渐近安全范式的基础核心思想，该范式拓展了可重整量子场论的适用范围。该范式极大程度借鉴了临界现象理论中涌现出的重整化群诠释，这与温伯格之前粒子物理学所采用的纯实用视角有显著区别。

The underlying assumption of the programs outlined so far is that renormaliz-ability has to take place close to the Gaußian fixed point of a theory. If we then desire that the finite theory is ultraviolet complete, the Gaußian fixed point must be ultraviolet and the theory asymptotically free. This surely looks like the most natural assumption when looking for UV completeness within the perturbative regime; however, asymptotic freedom is not the only possibility that may arise. In [86, 87] Weinberg suggested that an alternative to asymptotic freedom could be a scenario in which the couplings of the theory reach scale invariance at infinite energy and physical observables remain finite. In general, such regime might be located outside the validity of perturbation theory, even though in the special case of asymptotic freedom, it is captured by perturbative methods. This (generally nonperturbative) regime is referred to as asymptotic safety and is best phrased in terms of effective field theory methods (see [88] for a more recent account of the approach). Here we list the fundamental concepts and subtleties that become important when dealing with gravity following the original formulation by Weinberg [87].

迄今为止所梳理项目的基础假设是: 可重整化必须发生在理论的高斯不动点附近。如果我们要求有限的理论是紫外完备的，那么高斯不动点必须位于紫外区，且理论是渐近自由的。在微扰区域内寻找紫外完备性时，这无疑是最自然的假设；但渐近自由并非唯一可能的情况。在文献 [86, 87] 中，温伯格提出，渐近自由之外还存在一种替代方案: 理论的耦合在无限能量处达到标度不变性，且物理观测量始终保持有限。一般而言，这种机制可能处于微扰论的适用范围之外，尽管在渐近自由的特殊情况下可以用微扰方法处理。这种 (通常为非微扰的) 机制被称为渐近安全，用有效场论方法描述最为合适 (关于该方法的最新综述参见文献 [88])。本文我们将梳理遵循温伯格最初表述 [87]、研究引力问题时需要关注的核心概念与关键细节。

Let us consider an observable r associated with some physical process. In general, r will depend on some characteristic energy E of the process under consideration, the set of couplings of the underlying field theory g_i , and a set of dimensionless variables ζ which depend on the external momenta involved in the process. The expression for r can be cast as

我们考虑某物理过程对应的观测量 r 。一般来说，r 会依赖于该过程的特征能量 E 、基础场论的一组耦合 g_i ，以及一组依赖于过程涉及的外动量的无量纲变量 ζ 。r 的表达式可以写为

$$r = \mu^D f\left(\frac{E}{\mu}, \bar{g}_i(\mu), \zeta\right), \tag{47}$$

where D is the total mass dimension of r, μ is the renormalization scale and $\bar{g}_i(\mu) = \mu^{d_i} g_i(\mu)$ are dimensionless coupling constants. Making use of the fact that the observable should not depend on μ , we are free to set it to $\mu = E$ and obtain a simple scaling relation:

其中 D 是总质量维数，r, μ 是重整化标度，$\bar{g}_i(\mu) = \mu^{d_i} g_i(\mu)$ 是无量纲耦合常数。利用观测量不依赖于 μ 的性质，我们可以自由将其设为 $\mu = E$ ，得到简单的标度关系:

$$r = E^D f(1, \bar{g}_i(E), \zeta). \tag{48}$$

It is clear from this last equation that, aside from a trivial scaling E^D , the behavior of r is determined by that of the dimensionless couplings $\bar{g}(\mu)$ as $\mu \to \infty$. We write:

从上述方程可以清楚看出，除了平凡标度 E^D 之外，r 的行为由无量纲耦合 $\bar{g}(\mu)$ 在 $\mu \to \infty$ 下的行为决定。我们可以写出:

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \bar{g}_i(\mu) = \beta_i(\bar{g}_i(\mu)), \tag{49}$$

where, on the ground of dimensional arguments, the β functions cannot depend explicitly on μ (couplings are expressed in units of μ itself). Note that if the couplings reach a fixed point

其中，基于量纲分析，β 函数不能显依赖于 μ (耦合本身就以 μ 为单位)。注意若耦合达到不动点

$$\bar{g}_i(\mu) \underset{\mu \to \infty}{\to} \bar{g}_i^* \tag{50}$$

then the observable r has the simple scaling E^D , and the theory is free from unphysical divergences (the ratio between the observable and E^D is a finite prediction). The condition for scale invariance (50) is equivalent to the vanishing of the β -function:

则观测量 r 满足简单标度关系 E^D ，且理论不存在非物理发散 (观测量与 E^D 的比值是有限预言)。标度不变性条件 (50) 等价于 β β 函数等于零:

$$\beta_i(\bar{g}_i^*) = 0. \tag{51}$$

Note that the existence of a fixed point for the dimensionless couplings is not sufficient to determine a scale-invariant theory in the ultraviolet. The theory, in fact, needs to sit on a trajectory in the coupling space such that its renormalization group evolution hits the fixed point. The set of couplings that are dragged into the fixed point $\bar{g}_i^*$ when evolving toward the UV form the ultraviolet critical surface of fixed point.

注意，无量纲耦合存在不动点，并不足以保证紫外区存在标度不变的理论。实际上，理论需要位于耦合空间中满足如下条件的轨迹上: 其重整化群演化能够到达该不动点。当向紫外演化时，被拖入不动点 $\bar{g}_i^*$ 的一组耦合构成该不动点的紫外临界曲面。

A few comments are in order:

下面给出几点说明:

- The stress that we put on true physical observables is motivated by the fact that they do not depend on local field redefinitions. This requirement is equivalent to retain only essential couplings, i.e., to discard all those couplings γ for which a variation of the Lagrangian $\mathcal{L}(\varphi_n, \partial\varphi_n, \ldots)$ is proportional to the Euler-Lagrange equations of motion up to total derivative terms:

- 我们强调真实物理观测量的重要性，是因为它们不依赖于局域场重新定义。这个要求等价于只保留本质耦合，即丢弃所有满足下述条件的耦合 γ : 拉格朗日量 $\mathcal{L}(\varphi_n, \partial\varphi_n, \ldots)$ 的变分正比于欧拉-拉格朗日运动方程，差一个全导数项:

$$\frac{\partial \mathcal{L}}{\partial \gamma} \doteq \sum_n \left[\frac{\partial \mathcal{L}}{\partial \varphi_n} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi_n)} + \dots \right] F_n (\varphi_n, \partial \varphi_n, \dots), \tag{52}$$

where $\doteq$ means that total derivatives have been discarded. A coupling γ satisfying (52) is called inessential (a simple example would be the wavefunction renormalization of a self-interacting scalar). If inessential couplings are included, the system is not guaranteed to hit the fixed point under the renormalization group flow, even though the system is physically equivalent. The case of gravity is somewhat peculiar because Planck's mass requires special considerations [71].

其中 $\doteq$ 表示已舍去全导数项。满足式 (52) 的耦合 γ 被称为非本质耦合 (一个简单例子是自相互作用标量的波函数重整化)。若包含非本质耦合，即便系统在物理上等价，也无法保证重整化群流作用下系统会收敛到不动点。引力的情况比较特殊，因为普朗克质量需要特殊处理 [71]。

- The dimensionality of the ultraviolet critical surface gives us the number of free parameters that must be fine-tuned for the theory to reach scale invariance in the UV. It is therefore crucial that the critical surface be of finite dimension $\delta < \infty$; otherwise, the theory loses its predictive power. δ can be calculated perturbatively by linearizing the system of β -functions around the UV fixed point and is given by the number of negative eigenvalues of the stability matrix:

- 紫外临界曲面的维数即为了让理论在紫外达到标度不变性必须微调的自由参数数目。因此, 临界曲面必须是有限维的 $\delta < \infty$ ，否则该理论会失去预测能力。δ 可以通过将 β 函数系统在紫外不动点附近线性化做微扰计算，其结果等于稳定矩阵负本征值的数目:

$$S_{ij} = \left. \frac{\partial \beta_i}{\partial \bar{g}_j} \right|_{\bar{g}=\bar{g}^*} \tag{53}$$

- As in every approach to renormalization, a special care should be paid when "following" the flow toward the UV. Any Wilsonian approach to renormalization, which is based on the coarse-graining of some fluctuations, describes a flow toward the IR (it is a semigroup); therefore, UV fixed points can only be detected indirectly. In the present discussion, we are silently bypassing this issue and assume that a systematic analysis of the theory space underlies the analysis.

- 和所有重整化方法一样，在" 追踪" 流向紫外的流时需要格外小心。任何基于对涨落做粗粒化处理的威尔逊重整化方法，描述的都是流向红外的流 (它是半群)；因此，紫外不动点只能被间接探测到。在本次讨论中，我们默认跳过了这个问题，假设该分析建立在对理论空间的系统分析之上。

Gravity in $d = 2 + \varepsilon$

$d = 2 + \varepsilon$ 中的引力

We present the first practical computation with which the possibility that gravity is asymptotically safe was established. The results are based on the perturbative ε -expansion above $d = 2$ dimensions.

我们给出了首个实用计算，证明引力渐近安全的可能性成立。结果基于高于 $d = 2$ 维的微扰 ε 展开。

There are two important things happening to the Einstein-Hilbert action in the limit $d = 2$. Newton's constant becomes dimensionless, and the Einstein-Hilbert term becomes topological, because it is proportional to the Euler character:

在 $d = 2$ 极限下，爱因斯坦-希尔伯特作用量发生两个关键变化: 牛顿常数变为无量纲量，且爱因斯坦-希尔伯特项成为拓扑项，因为它正比于欧拉示性数:

$$\chi = \int \mathrm{d}^2 x \sqrt{g} R \tag{54}$$

The left-hand side of Einstein's equations therefore vanishes identically (thanks to the contracted differential Bianchi identities).

因此，爱因斯坦方程的左侧恒等于零 (得益于缩并的微分比安基恒等式)。

In general d , Newton's coupling constant has dimensions $[G] = 2-d$ and becomes dimensionless for $d = 2$. Using dimensional regularization, one is therefore tempted to discard possible countertems proportional to the Lagrangian, on the ground that χ vanishes for $\varepsilon = d - 2$ going to zero (We implicitly assume that the topology of the spacetime is that of a non-compact, asymptotically flat manifold; otherwise χ would have a finite value and possible divergences of the form χ/ε would have no reason to be ignored in the limit $\varepsilon \to 0$). As it turns out, $\varepsilon = 0$ is a nonanalytic limit for the Green functions, implying that the limit $\varepsilon \to 0$ does not coincide with the theory defined in exactly two dimensions. Hence, for dimensional regularization to make sense, one is forced to define two-dimensional gravity as the limit of $(2 + \varepsilon)$ -dimensional gravity, and counterterms for the Lagrangian are therefore necessary even if it is topological.

一般情况下, 在 d 中牛顿耦合常数具有量纲 $[G] = 2-d$, 在 $d = 2$ 时变为无量纲量。因此在维数正则化中，人们很容易因为当 $\varepsilon = d-2$ 趋于零时 χ 趋于零，就舍弃拉格朗日对应的可能 counterterm(我们隐含假设时空拓扑是非紧致渐近平直流形；否则 χ 会取有限值，形如 χ/ε 的可能发散在 $\varepsilon \to 0$ 极限下没有理由被忽略)。结果表明，$\varepsilon = 0$ 对格林函数而言是非解析极限，意味着 $\varepsilon \to 0$ 极限与精确二维下定义的理论并不一致。因此，为了让维数正则化有意义，我们必须将二维引力定义为 $(2+\varepsilon)$ 维引力的极限，即使拉格朗日是拓扑性的，其 counterterm 仍然是必要的。

Let us start by isolating the analytic part of the unrenormalized Newton's constant G_0 for finite ε :

我们首先分离出有限 ε 下未重整化牛顿常数 G_0 的解析部分:

$$G_0(\varepsilon)\mu^{\varepsilon} = \bar{G}(\mu) + \sum_n b_n\left(\bar{G}(\mu)\right)\varepsilon^{-n}, \tag{55}$$

with renormalization group equation

满足重整化群方程

$$\mu\frac{\mathrm{d}}{\mathrm{d}\mu}G(\mu) = \beta(G(\mu), \varepsilon), \tag{56}$$

where we dropped the bar for the dimensionless Newton's constant. At leading order, one finds the beta function:

其中我们省略了无量纲牛顿常数的上划线。领头阶下可得 beta 函数:

$$\beta(G,\varepsilon) = \varepsilon G + b_1(G) - G\frac{\partial b_1}{\partial G}. \tag{57}$$

Further expanding the beta function for small values of G , we can expect the coefficient b_1 to have the generic structure:

进一步对小 G 展开 beta 函数，我们可以预期系数 b_1 具有一般结构:

$$b_1(G) = bG^2 + o\left(G^3\right), \tag{58}$$

leading to

由此得到

$$\beta(G,\varepsilon) = \varepsilon G - bG^2. \tag{59}$$

We notice that, if the coefficient b is positive, the β -function for G has two fixed points: one corresponds to the Gaußian theory for vanishing Newton's coupling, while the other occurs for the value $G^* = \varepsilon/b$. Inspecting the form of Eq. (59), we notice that G^* is an ultraviolet fixed point of the renormalization group flow which constitute a viable candidate for an asymptotically safe theory of gravity at finite ε close to two dimensions. The value of the fixed point is perturbative as long as one treats ε as a small parameter, bypassing a lot of issues concerning nonperturbative computations. Moreover, as we send ε to zero and approach $d = 2$ dimensions, G^* decreases until it collides with the Gaußian fixed point, and the theory becomes asymptotically free.

我们注意到，如果系数 b 为正，G 对应的 β 函数存在两个不动点: 一个对应牛顿耦合为零的高斯理论，另一个出现在数值 $G^* = \varepsilon/b$ 处。考察式 (59) 的形式可知，G^* 是重整化群流的紫外不动点，它是接近二维的有限 ε 处引力渐近安全理论的合理候选。只要我们将 ε 视作小参数，不动点的数值就是微扰的，绕开了很多非微扰计算相关的问题。此外，当我们让 ε 趋于零、趋近 $d = 2$ 维时，G^* 会减小直到与高斯不动点碰撞，理论变为渐近自由的。

The key feature for such a scenario to hold is that the coefficient b is positive. At one loop, integrating the fluctuations of the metric with a linear splitting over a background (see sections "Appendix A: Background Field Method" and "Appendix B: Covariant Computations and the Heat Kernel" for more details on the general procedures) and quantum matter fluctuations, one finds the value [12, 14, 15, 84, 87] :

该情景成立的核心条件是系数 b 为正。一圈阶下，对背景上线性拆分得到的度规涨落 (一般步骤的更多细节参见附录 A: 背景场方法和附录 B: 协变计算与热核) 和量子物质涨落做积分，可得数值 [12, 14, 15, 84, 87] :

$$b = \frac{2}{3}\left(19 + 6N_V - \frac{1}{2}N_F - N_S\right). \tag{60}$$

where N_V is the number of gauge fields coupled to the theory, N_F the number of Majorana fermions and N_S the number of scalar fields. One can see that the fixed point G^* exists for pure gravity, but also as long as the number of vector bosons is enough to compensate the presence of fermions and scalars.

其中 N_V 是与理论耦合的规范场数目，N_F 是马约拉纳费米子数目，N_S 是标量场数目。可以看到，纯引力存在不动点 G^*；只要矢量玻色子的数目足够抵消费米子和标量的影响，该不动点也仍然存在。

Before buying into the asymptotic safety scenario, however, we need to pay special attention to the physical interpretation of our result. In a purely gravitational setting, Einstein equations set $R = 0$ and Newton' s coupling turns out to be inessential, and there is no need for it to reach a fixed point as μ grows to infinity. Upon the introduction of matter fields, one can use Einstein equations to express R in terms of the trace of the energy momentum tensor; therefore, G is not an independent coupling. However, the situation changes once we introduce new interactions to compare $\sqrt{g}R$ to. Two common choices are given by Gibbons-Hawking-York's boundary operator [38,91,92], which is the most natural in the case of a manifold with boundary and was used in [37] to define the essential coupling of two-dimensional gravity, and by the cosmological constant, which is useful in the absence of boundary as shown in [16]. One can fix either of these two operators using Einstein equations and extract the flow for an essential coupling expressed in terms of the curvature. For example, one can explicitly check that the combination $\widetilde{G} = G_N \Lambda^{\frac{d-2}{2}}$ turns out to have a β -function which does not depend on the gauge parameters and, at one loop in $d = 2 + \varepsilon$, reads [45]:

然而，在接受渐近安全方案之前，我们需要特别关注我们结果的物理解释。在纯引力情形下，爱因斯坦方程确定了 $R = 0$ ，牛顿耦合并不本质，无需随着 μ 趋于无穷而达到不动点。引入物质场后，我们可以利用爱因斯坦方程将 R 用能量动量张量的迹表示；因此 G 并非独立耦合。但当我们引入新的相互作用来和 $\sqrt{g}R$ 对比时，情况发生了改变。两种常见选择分别是吉本斯-霍金-约克边界算符 [38,91,92] 与宇宙学常数: 前者对带边界流形是最自然的，文献 [37] 用它定义了二维引力的本质耦合；后者如文献 [16] 所示，在无边界的情形下十分有用。我们可以通过爱因斯坦方程固定这两种算符中的任意一种，得到以曲率表示的本质耦合的流。例如，可以明确验证，组合 $\widetilde{G} = G_N \Lambda^{\frac{d-2}{2}}$ 的 β 函数不依赖规范参数，且在 $d = 2 + \varepsilon$ 的一阶圈修正下，其形式为 [45]:

$$\frac{1}{(4\pi)^2} \beta_{\widetilde{G}} = -\frac{2}{3} 19 \widetilde{G}^2. \tag{61}$$

Note that, while in $d = 2$ this combination reduces to just G_N , in $d = 4$ this coincides with the coupling γ in Eqs. (23) and (32).

注意，尽管在 $d = 2$ 中该组合仅约化为 G_N ， 在 $d = 4$ 中它与式 (23) 和 (32) 中的耦合 γ 一致。

The Story of the Conformal Mode in $2 + \varepsilon$ Dimensions

$2 + \varepsilon$ 维中共形模的故事

We discuss subtleties of the perturbative analysis in $2 + \varepsilon$ dimensions that involve the conformal mode as they emerged in the original literature. These considerations were pivotal in pushing the analysis to two loops, because a nonlinear parametrization of the conformal mode of the metric was needed.

我们讨论原始文献中提出的、$2+\varepsilon$ 维微扰分析里涉及共形模的精妙问题。这些考量对将分析推进到两圈阶至关重要，因为需要对度规的共形模进行非线性参数化。

The application of dimensional regularization to two-dimensional gravity was pursued with some dedication in a series of papers with the aim to go beyond the leading order discussed in section "Gravity in $d = 2+\varepsilon$". The seminal attempts are from Kawai and Ninomiya [49], and from Jack and Jones [53], who tried to renormalize the Einstein-Hilbert theory at two loops in $d = 2+\varepsilon$ dimensions with cosmological constant. Notice that the original work by Jack and Jones actually studies gravity in $d = 2-\varepsilon$. It is worth pointing out that, formally, Feynman diagrams are indeed convergent below $d = 2$ rather than above. Moreover, the region $1 < d < 2$ has the advantage of being free of conformal instability. It is simple to formally continue the theory above $d = 2$ after regularizing Feynman integrals, though the continuation may encounter nonperturbative problems [61]. Here we just present everything in $d = 2+\varepsilon$ for a better comparison among all works in the literature. The seminal papers [49] and [53] draw very different conclusions on the renormalizability of gravity close to two dimensions. We present the historical solution of the conundrum here, while we reserve our modern take for section "Revisiting Quantum Gravity in $2+\varepsilon$".

人们投入了不少精力在一系列论文中将维数正规化应用到二维引力，旨在超越"$d=2+\varepsilon$ 维引力"一节中讨论的领头阶。川合 (Kawai) 和二宫 (Ninomiya)[49] 以及杰克 (Jack) 和琼斯 (Jones)[53] 做了开创性尝试，他们试图对带宇宙常数的 $d=2+\varepsilon$ 维爱因斯坦-希尔伯特理论做两圈重整化。请注意杰克和琼斯的原始工作实际研究的是 $d=2-\varepsilon$ 维引力。值得指出，形式上费曼图确实在低于 $d=2$ 维时收敛，而非高于 $d=2$ 维。此外，区域 $1<d<2$ 没有共形不稳定性的优势。对费曼积分正规化后，可以很容易地将理论形式解析延拓到 $d=2$ 维以上，不过解析延拓可能会遇到非微扰问题 [61]。为了更好地比较文献中的所有工作，本文统一采用 $d=2+\varepsilon$ 维表述。开创性论文 [49] 和 [53] 对近二维引力的可重整性得出了截然不同的结论。我们在此给出这个谜题的历史性解答，而我们的现代观点放在"重访 $2+\varepsilon$ 维量子引力"一节。

The origin of the hurdle is in Ref. [53], where the authors point out possible issues with dimensional poles, which, in the authors' interpretation, hint that gravity is not two-loop renormalizable in $d = 2+\varepsilon$. In short, the problem is caused by the mixing of the dimensional poles coming from regulating Feynman diagrams with the kinematical poles coming from the fact that Einstein-Hilbert gravity is topological in $d = 2$ (so the number of degrees of freedom changes discontinuously). After that, in an impressive series of papers [1-3,36,46-48,69], Kawai and many collaborators improved on the theoretical aspects of gravity close to two dimensions and its renormalization properties following the original results of [49]. Ultimately, these efforts led to the result of Aida and Kitazawa [3] where the renormalizability of quantum gravity in $d = 2+\varepsilon$ in the conformal gauge was proved to two-loop and the critical exponents of the theory were computed.

障碍起源于文献 [53]，作者在其中指出维数极点可能存在问题，在作者的解读中，这暗示引力在 $d=2+\varepsilon$ 维下不是两圈可重整的。简单来说，问题的成因是: 正规化费曼图带来的维数极点，与爱因斯坦-希尔伯特引力在 $d=2$ 维是拓扑理论 (因此自由度数目发生不连续变化) 带来的运动极点发生了混合。在此之后，川合和众多合作者在一系列重磅论文 [1-3,36,46-48,69] 中，基于 [49] 的原始结果改进了近二维引力的理论方面及其重整化性质。这些工作最终得到了相田 (Aida) 和北泽 (Kitazawa)[3] 的结果，该结果证明了共形规范下 $d=2+\varepsilon$ 维量子引力到两圈阶都是可重整的，并计算了理论的临界指数。

This section gives a concise review of the difficulties encountered by Jack and Jones and of the results obtained in the conformal gauge, mainly following [1,3]. Some considerations will play an important role in the analysis of section ”Revisiting Quantum Gravity in $2+\varepsilon$ ”, which instead comes from more recent works [39, 60]. We start by reviewing the peculiar importance of the conformal mode and its relation to the fact that the number of degrees of freedom propagated by the Einstein-Hilbert action has a discontinuity in $d=2$.

本节简要回顾杰克和琼斯遇到的困难，以及在共形规范下得到的结果，主要遵循文献 [1,3] 的表述。部分结论会在“重访 $2+\varepsilon$ 维量子引力”一节的分析中发挥重要作用，那一节内容来自更新的研究 [39, 60]。我们首先回顾共形模的特殊意义，以及它和爱因斯坦-希尔伯特作用量传播的自由度数目在 $d=2$ 维处发生不连续变化这一事实的关联。

We start by following [53] and considering the action:

我们首先遵循文献 [53]，考虑如下作用量:

$$S[g]=-\int \mathrm{d}^d x\sqrt{g}\left(\frac{Z_G}{G}R-Z_\Lambda\Lambda\right), \tag{62}$$

where Z_G and Z_Λ are renormalization constants. Consider now a splitting of the metric in a background and fluctuations as

其中 Z_G 和 Z_Λ 是重整化常数。现在将度规分解为背景部分和涨落部分，写作

$$g_{\mu\nu}=Z_g\left(\bar{g}_{\mu\nu}+Z_h\sqrt{G}h_{\mu\nu}\right), \tag{63}$$

in which Z_g and Z_h are wavefunction renormalization for background and fluctuations, respectively. Diffeomorphisms are gauge-fixed using the Feynman-de Donder gauge-fixing action:

其中 Z_g 和 Z_h 分别是背景和涨落的波函数重整化。我们使用费曼-德东德规范固定作用量来固定微分同胚规范:

$$S_{gf}[h;\bar{g}]=\frac{1}{2}\int \mathrm{d}^d x\sqrt{\bar{g}}\bar{g}^{\mu\nu}F_\mu F_\nu, \tag{64}$$

$$F_\mu=\overline{\nabla}_\alpha h^\alpha_\mu-\frac{1}{2}\overline{\nabla}_\mu h^\alpha_\alpha.$$

The part of the gauge-fixed action that is quadratic in the fluctuations and covariant over the background, i.e., the Hessian, is

规范固定作用量中涨落二次项、且对背景协变的部分，即黑塞矩阵，为

$$S^{(2)}[h;\bar{g}]=\frac{1}{2}\int \mathrm{d}^d x\sqrt{\bar{g}}h_{\mu\nu}\left(-\overline{\nabla}^2K^{\mu\nu\alpha\beta}+X^{\mu\nu\alpha\beta}\right)h_{\alpha\beta}, \tag{65}$$

where $\overline{\nabla}^2=\overline{\nabla}^\mu\overline{\nabla}_\mu$ is the background compatible covariant Laplacian. The endomorphism X is a function of background curvature operators, and the matrix K is

其中 $\overline{\nabla}^2 = \overline{\nabla}^\mu \overline{\nabla}_\mu$ 是适配背景的协变拉普拉斯算符。自同态 X 是背景曲率算符的函数，矩阵 K 为

$$K^{\mu\upsilon\alpha\beta} = \frac{1}{2}\left(\bar{g}^{\mu\alpha}\bar{g}^{\upsilon\beta} + \bar{g}^{\mu\beta}\bar{g}^{\upsilon\alpha} - \bar{g}^{\mu\upsilon}\bar{g}^{\alpha\beta}\right). \tag{66}$$

The differential operator appearing in (65) is not invertible in $d = 2$ because the matrix K is not invertible. In general d this problem takes the form of a divergent term for $d \sim 2$ in the inverse

(65) 式中出现的微分算符在 $d = 2$ 维下不可逆，因为矩阵 K 不可逆。一般来说，在 d 情况下，这个问题在逆算符中表现为 $d \sim 2$ 的发散项

$$K^{-1}_{\mu\upsilon\alpha\beta} = \frac{1}{2}\left(\bar{g}_{\mu\alpha}\bar{g}_{\upsilon\beta} + \bar{g}_{\mu\beta}\bar{g}_{\upsilon\alpha} - \frac{1}{d-2}\bar{g}_{\mu\upsilon}\bar{g}_{\alpha\beta}\right). \tag{67}$$

The pole in K^{-1} is of kinematical origin because it is caused by the fact that in (and only in) $d = 2$ the only degree of freedom of the metric that can propagate is the conformal one.

K^{-1} 中的极点起源于运动学，因为它由以下事实导致: 在 (且仅在) $d = 2$ 中，度规唯一可传播的自由度就是共形自由度。

However, before taking the limit $\varepsilon \to 0$, we could still apply the heat kernel techniques to compute the effective action to the Hessian (65) multiplied by K^{-1} , i.e., consider the kinetic operator:

然而，在取 $\varepsilon \to 0$ 极限之前，我们仍可以应用热核技术来计算乘以 K^{-1} 后的黑塞矩阵 (65) 的有效作用量，即考虑如下动能算符:

$$-\nabla^2 I_{\mu\nu\alpha\beta} + K^{-1}_{\mu\nu\rho\sigma} X^{\rho\sigma}_{\alpha\beta}, \tag{68}$$

that is finite for $d \sim 2$, given I the identity on the space of symmetric tensors. This operation can be thought of as a change of variable with unit Jacobian in the path integral, which should not affect the one-loop computation of the effective action; however, divergences of higher loop diagrams are potentially sensitive to the redefinition caused by K^{-1} . In fact, it seems that all the propagators of loop diagrams will come with an extra (kinematical) ε -pole that must combine with similar poles coming from dimensional regularization. The authors of Ref. [53] then deduce that subleading divergences will fail to cancel (the cancellation of subleading divergences beyond one loop is always a delicate point; specifically they show that one cannot find a wavefunction renormalization that cancels the nonlocal divergences at two loops).

给定 I 是对称张量空间上的恒等算符，该算符对 $d \sim 2$ 是有限的。这个操作可以被理解为路径积分中雅可比行列式为 1 的变量替换，它不会影响有效作用量的单圈计算；但高圈图的发散可能会对 K^{-1} 导致的重新定义敏感。实际上，所有圈图的传播子都会额外携带一个 (运动学的) ε 极点，它必须和维数正则化产生的类似极点结合。文献 [53] 的作者由此推断次 leading 发散无法抵消 (单圈以外的次 leading 发散抵消本身就是一个微妙问题；他们具体证明了不存在能抵消两圈非局域发散的波函数重整化)。

However, the presence of the kinematical pole does not find its origin in the dimensional regularization method, rather in the symmetries of the original theory and in the topological nature of the scalar curvature in $d=2$. One can see that the role of the $(d-2)^{-1}$ term is to give an infinite weight to the trace mode of the fluctuations as d approaches 2 and to flip the sign of the kinetic term for such mode.

但这个运动学极点并非起源于维数正则化方法，而是源于原始理论的对称性，以及 $d=2$ 中标量曲率的拓扑性质。可以看出，当 d 趋近于 2 时，$(d-2)^{-1}$ 项的作用是给涨落的迹模式赋予无穷大权重，并翻转该模式动能项的符号。

To better understand the situation, let us consider the Einstein-Hilbert action without cosmological constant and parametrize the full metric, which we now refer to as $\bar{g}$, in terms of a fiducial metric $\hat{g}$ and the conformal mode ϕ as

为了更好地理解这个情况，我们考虑没有宇宙学常数的爱因斯坦-希尔伯特作用量，将我们这里记为 $\bar{g}$ 的 full 度规用参考度规 $\hat{g}$ 和共形模式 ϕ 参数化为:

$$g_{\mu\nu}=\hat{g}_{\mu\nu}e^{-\phi}. \tag{69}$$

One gets:

得到:

$$\int \mathrm{d}^d x\sqrt{g}R=\int \mathrm{d}^d x\sqrt{\hat{g}}e^{-\frac{\varepsilon}{2}\phi}\left[\hat{R}-\varepsilon\frac{d-1}{4}\hat{g}^{\mu\upsilon}\partial_\mu\phi\partial_\upsilon\phi\right], \tag{70}$$

from which is manifest that the action is Weyl invariant in the limit $\varepsilon\to 0$ as expected by its the topological nature. However, if we now parametrize the conformal mode as

从中可以清楚看出，作用量在 $\varepsilon\to 0$ 极限下具有外尔不变性，这符合它的拓扑性质的预期。如果我们现在将共形模式参数化为:

$$e^{-\frac{\varepsilon}{4}\phi}=\sqrt{\frac{\varepsilon}{8(d-1)}}\psi, \tag{71}$$

it is easy to see that the original action is equivalent to the theory of a scalar conformally coupled to the background geometry:

很容易看出原始作用量等价于一个与背景几何共形耦合的标量场论:

$$\int \mathrm{d}^d x\sqrt{g}R=\int \mathrm{d}^d x\sqrt{\hat{g}}\left[\hat{R}\frac{\varepsilon}{8(d-1)}\psi^2-\frac{1}{2}\hat{g}^{\mu\upsilon}\partial_\mu\psi\partial_\upsilon\psi\right]. \tag{72}$$

We can then try to quantize the latter action under the prescription of preserving Weyl invariance. This procedure involves a rescaling in the dilaton mode that is singular as seen in (71) (The interpretation reviewed in section ”Revisiting Quantum Gravity in $2+\varepsilon$ ” solves the problem in an entirely different way.). Note that the domain of the field ψ is classically constrained to be positive, according to Eq. (71), but in the quantization procedure, all fluctuations are allowed.

我们接下来可以尝试在保持外尔不变性的规则下量子化这个作用量。这个过程涉及 dilaton 模式的重标度，从 (71) 可以看出该重标度是奇异的 (“$2+\varepsilon$ 中再探量子引力”一节回顾的解释用一种完全不同的方式解决了这个问题)。注意根据式 (71)，经典下场 ψ 的取值被约束为正，但在量子化过程中所有涨落都是允许的。

An important point to keep in mind is that we are enlarging the symmetry group of our theory by including Weyl symmetry. However, it so happens that, locally, the diffeomorphisms group $\mathrm{Diff}(\mathcal{M})$ of the spacetime manifold $\mathcal{M}$ is isomorphic to the semidirect product $\mathrm{Diff}^* = \mathrm{SDiff}(\mathcal{M}) \ltimes \mathrm{Weyl}$, where SDiff is the group of volume-preserving diffeomorphisms, while Weyl is the group of Weyl transformation of the metric. In terms of the classical fields $\hat{g}$ and ϕ , Diff* acts as

需要牢记的一个要点是: 我们通过引入外尔对称性扩大了理论的对称群。但碰巧的是，局部上来看，时空流形 $\mathcal{M}$ 的微分同胚群 $\mathrm{Diff}(\mathcal{M})$ 同构于半直积 $\mathrm{SDiff}^* = \mathrm{SDiff}(\mathcal{M}) \ltimes \mathrm{Weyl}$，其中 SDiff 是保体积微分同胚群，Weyl 是度规的外尔变换群。用经典场 $\hat{g}$ 和 ϕ 表示，Diff* 的作用为:

$$\delta^*_\xi \hat{g}_{\mu\upsilon} = \mathcal{L}_\xi \hat{g}_{\mu\upsilon} - \frac{2}{d}\hat{g}_{\mu\upsilon}\hat{\nabla}_\rho \xi^\rho,\ \delta^*_\xi \phi = \mathcal{L}_\xi \phi + \frac{\varepsilon}{2d}\phi\hat{\nabla}_\rho \xi^\rho, \tag{73}$$

where $\mathcal{L}_\xi$ is the Lie derivative along the vector field ξ and $\hat{\nabla}$ is the covariant derivative compatible with $\hat{g}$. In the background field method, one can break $\mathrm{Diff}(\mathcal{M}) \ltimes \mathrm{Weyl}$ to Diff* by splitting the fiducial metric as

其中 $\mathcal{L}_\xi$ 是沿向量场 ξ 的李导数，$\hat{\nabla}$ 是与 $\hat{g}$ 相容的协变导数。在背景场方法中，我们可通过将参考度规分解为，将 $\mathrm{Diff}(\mathcal{M}) \ltimes \mathrm{Weyl}$ 对称性破缺为 Diff*

$$\hat{g}_{\mu\nu} = \bar{g}_{\mu\rho}\left(e^h\right)^\rho_\nu \tag{74}$$

and requiring that the fluctuation field is traceless $g^{\mu\upsilon}h_{\mu\upsilon} = 0$ so that $\det\hat{g} = \det\bar{g}$. One then expands the scalar field around a constant vev as

并要求涨落场是无迹的 $g^{\mu\upsilon}h_{\mu\upsilon} = 0$ ，使得 $\det\hat{g} = \det\bar{g}$ 。随后我们将标量场围绕常数真空期望值展开:

$$\sqrt{\frac{\varepsilon}{8(d-1)}}\psi \to \mu^{\frac{\varepsilon}{4}}\left(1 + \frac{1}{2}\sqrt{\frac{\varepsilon}{2(d-1)}}\psi\right), \tag{75}$$

and considers the gauge-fixing action

并考虑规范固定作用量

$$S_{gf}[h,\psi;\bar{g}] = \frac{1}{2}\int \mathrm{d}^d x\sqrt{\bar{g}}\bar{g}^{\mu\upsilon}F_\mu F_\upsilon,$$

$$F_\mu = \overline{\nabla}^\alpha h_{\alpha\mu} - \sqrt{\frac{\varepsilon}{2(d-1)}}\partial_\mu\psi \tag{76}$$

Upon including the ghost action in the usual way, we can compute the one-loop divergence for the effective action of the theory and obtain

按照常规方式引入鬼作用量后，我们可以计算该理论有效作用量的单圈发散，得到

$$\Gamma_\infty = \frac{25}{24\pi}\frac{\mu^\varepsilon}{\varepsilon}\int \mathrm{d}^d x\sqrt{\bar{g}}\bar{R}. \tag{77}$$

Equation (77) breaks Weyl invariance and makes the quantum theory anomalous.

式 (77) 破坏了外尔不变性，导致量子理论出现反常

At this point one can resort on two strategies to obtain a Weyl symmetric quantum theory. The first option is to add matter degrees of freedom hoping that the corresponding counterterms can be used to cancel the anomaly. In this approach the most popular choice is to include conformally coupled scalar fields φ_i at tree level as

此时可以采用两种方案得到外尔对称的量子理论。第一种方案是引入物质自由度，期望利用对应的抵消项消去反常。该方法中最常用的选择是在树图阶引入共形耦合标量场 φ_i ，形式为

(78)

$$S = \frac{\mu^\varepsilon}{G}\int \mathrm{d}^d x\sqrt{\bar{g}}\left\{\hat{R}\left[\left(1+\frac{1}{2}\sqrt{\frac{\varepsilon}{2(d-1)}}\psi\right)^2 - \frac{\varepsilon}{8(d-1)}\varphi_i^2\right]\right.$$
$$\left. -\frac{1}{2}\partial_\mu\psi\partial_\upsilon\psi\hat{g}^{\mu\upsilon} + \frac{1}{2}\sum_i \partial_\mu\varphi_i\partial_\upsilon\varphi_i\hat{g}^{\mu\upsilon}\right\}.$$

Equation (77) turns into

式 (77) 变为

$$\Gamma_\infty = \frac{25-c}{24\pi}\frac{\mu^\varepsilon}{\varepsilon}\int \mathrm{d}^d x\sqrt{\bar{g}}\bar{R}, \tag{79}$$

where c is the number of scalar fields. One can then make the theory anomaly free by choosing $c = 25$. This way of restoring Weyl symmetry has been popularized by string theory [74, 75] , where the scalar multiplet $\{\psi, \varphi_i\}$ is interpreted as the embedding map of the string into the target space. In the string's language, the coefficient $25 - c$ is actually rewritten as $26 - (c + 1)$, where 1 is for the string’s conformal mode, which does not propagate over Polyakov's string, and 26 is the critical dimension of the string's target space.

其中 c 是标量场的数量。我们可以通过选择 $c = 25$ 使理论无反常。这种恢复外尔对称性的方法由弦理论推广而来 [74, 75] ，在弦理论中标量多重态 $\{\psi, \varphi_i\}$ 被解释为弦嵌入目标空间的映射。在弦的语言中，系数 $25 - c$ 实际上可以重写为 $26 - (c + 1)$ ，其中 1 对应弦的共形模式 (它不在波利亚科夫弦上传播)，26 对应弦目标空间的临界维数。

Alternatively, the cancellation of the anomaly can be achieved if we include in the bare action a term of the form

另一种方法是，如果我们在裸作用量中加入一项形式如下的项，也可以实现反常的消除

$$S_t = q \int \mathrm{d}^d x \sqrt{\bar{g}} \hat{R} \psi. \tag{80}$$

The constant q is known as the ”topological” charge and does not renormalize in (and only in) $d = 2$ [61]. Hence, even though q breaks Weyl symmetry at tree level, we are free to choose it so that it cancels the quantum anomaly. We return on these aspects of the conformal gauge of two-dimensional gravity in section ”Revisiting Quantum Gravity in $2 + \varepsilon$ ”.

常数 q 被称为“拓扑荷”，它仅在 $d = 2$ 中不发生重整化 [61]。因此，即使 q 在树图阶破坏外尔对称性，我们仍可以自由选取它来抵消量子反常。我们会在“重访 $2 + \varepsilon$ 中的量子引力”一节回到二维引力共形规范的这些相关讨论。

Two-Dimensional Gravity at Two Loops

两圈二维引力

The leading loop computation is not enough to guarantee the renormalizability of the theory because the cancellation of higher loops is delicate for the case of gravity. The two-loop computation for gravity in $d = 2 + \varepsilon$ dimensions using the rescaling of the conformal mode was performed by Aida and Kitazawa [3]. Given the uniqueness and the technical complexity of the two-loop computation, we recommend that it is repeated in the future by some courageous colleague.

领头阶圈图计算不足以保证该理论的可重整性，因为引力的高阶圈图抵消过程十分复杂。Aida 和 Kitazawa 利用共形模重标度完成了 $d = 2 + \varepsilon$ 维引力的两圈计算 [3]。鉴于该两圈计算的独特性与技术复杂度，我们建议未来能有勇敢的同行重复这项工作。

For completeness, Aida and Kitazawa consider a system of gravity coupled to matter fields and study the renormalization of the gravitational sector:

为求完备，Aida 和 Kitazawa 研究了引力耦合物质场的体系，并分析了引力子域的重整化:

$$S_{\text{grav}} = \frac{\mu^\varepsilon}{G} \int \mathrm{d}^d x \sqrt{\hat{g}} \left\{ \hat{R} \left(1 + a\psi + \varepsilon b \psi^2 \right) - \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \psi \partial_\nu \psi \right\},$$

$$S_{\text{matter}} = \frac{\mu^\varepsilon}{G} \int \mathrm{d}^d x \sqrt{\hat{g}} \sum_i \left\{ \frac{1}{2} \hat{g}^{\mu\nu} \partial_\mu \varphi_i \partial_\upsilon \varphi_i - \varepsilon b \varphi_i^2 \hat{R} \right\}, \tag{81}$$

where we adopted the same conventions as the previous section for the parametrization of metric degrees of freedom. Classically, the conformal values of the couplings a and b are given by $a^2 = 4\varepsilon b = \varepsilon/2\,(d-1)$ and are understood as the bare values of the microscopic (bare) theory. At one loop, a receives radiative corrections as long as $d \neq 2$ and has a Gaußian fixed point. Here, the gauge symmetry of the fields reads:

其中我们沿用了上一节对度规自由度参数化的相同约定。经典层面上，耦合常数 a 和 b 的共形值由 $a^2 = 4\varepsilon b = \varepsilon/2\,(d-1)$ 给出，它们对应微观裸理论的裸参数值。单圈层面，只要满足 $d \neq 2$，a 就会得到辐射修正，并存在一个高斯不动点。此处，场的规范对称性为:

$$\delta_\xi^* \hat{g}_{\mu\upsilon} = \mathcal{L}_\xi \hat{g}_{\mu\upsilon} - \frac{2}{d}\hat{g}_{\mu\upsilon}\hat{\nabla}_\rho \xi^\rho,$$

$$\delta_\xi^* \psi = \mathcal{L}_\xi \psi + \frac{2}{d}\left[(d-1)\,a + \frac{\varepsilon}{4}\psi\right]\hat{\nabla}_\rho \xi^\rho, \tag{82}$$

$$\delta_\xi^* \varphi_i = \mathcal{L}_\xi \varphi_i + \frac{2}{d}\left(\frac{\varepsilon}{4}\varphi_i\right)\hat{\nabla}_\rho \xi^\rho.$$

The gauge-fixing action in this parametrization can be obtained as a slight modification of (76)

该参数化下的规范固定作用量可通过对 (76) 式稍作修改得到

$$S_{gf} = \frac{\mu^\varepsilon}{G}\int \mathrm{d}^d x\sqrt{\bar{g}}\frac{1}{2}\left(\bar{\nabla}_\mu h_\upsilon^\mu - a\partial_\upsilon\psi\right)\left(\bar{\nabla}_\rho h^{\rho\upsilon} - a\partial^\upsilon\psi\right). \tag{83}$$

Evaluating the two-loop diagrams of the theory, one can explicitly observe the cancellation of $\frac{1}{\varepsilon^2}$ poles. In diagrams with two loops, one finds divergences that are nonlocal in the sense that they depend on global aspects of the background manifold through subleading terms of the propagator (see Eq. (133) in section ”Appendix B: Covariant Computations and the Heat Kernel”); however, most of these divergence-sare canceled by diagrams containing the one-loop counterterms, the only surviving one being proportional to the equations of motion. The total divergence of the theory is given by [3]

计算该理论的两圈费曼图后，可以清晰观测到 $\frac{1}{\varepsilon^2}$ 极点的抵消。在两圈图中，我们会发现依赖背景流形整体性质的非局域发散，这类发散来自传播子的次领头项 (详见“附录 B: 协变计算与热核”一节的式 (133))；但这类发散大部分都会被包含单圈抵消项的图抵消，仅有一项正比于运动方程的发散留存下来。该理论的总发散由文献 [3] 给出:

$$\Gamma_\infty = \frac{G}{(4\pi)^2}\int \mathrm{d}^d x\sqrt{\bar{g}}\frac{-79+3c}{8\varepsilon}\bar{R}$$
$$+\frac{G}{4\pi}\int \mathrm{d}^d x\sqrt{\bar{g}}\left(-\frac{11}{12\varepsilon}\right)\left(\bar{R}_{\mu\nu} - \frac{1}{2}\bar{R}\bar{g}_{\mu\nu}\right)\bar{G}^\mu{}_\rho{}^{\nu\rho}, \tag{84}$$

where $\bar{G}^{\mu\upsilon\rho\sigma}$ is defined through the Seeley-de Witt expansion of the graviton propagator (see Eqs. (129) and (133) in section ”Appendix B: Covariant Computations and the Heat Kernel”) as

其中 $\bar{G}^{\mu\upsilon\rho\sigma}$ 由引力子传播子的 Seeley-de Witt 展开定义 (详见“附录 B: 协变计算与热核”一节的式 (129) 和 (133)):

$$G^{\mu\nu\rho\sigma}(x,x') = G_0(x,x')\,a_0^{\mu\nu\rho\sigma}(x,x') + G_1(x,x')\,a_1^{\mu\nu\rho\sigma}(x,x') + \bar{G}^{\mu\nu\rho\sigma}.$$

(85)

$\bar{G}$ is regular in the coincident limit but depends on global aspects of the manifold, and its cancellation from physical divergences is crucial for the renormalization program. In our case $\bar{G}$ only appears multiplying Einstein's tensor meaning that it can be eliminated by going on-shell on the background geometry (in the limit $\varepsilon \to 0$) . Alternatively, as shown in [1, 3] , one can remove the nonlocal two-loop divergence with a suitable wavefunction renormalization Z_h with $O\left(\varepsilon^{-1}\right)$ poles. In this case, the parametrization of Z_h requires nonlinear terms in the fluctuations $h_{\mu\upsilon}$. The final two-loop divergence of gravity without cosmological constant in the conformal gauge is

$\bar{G}$ 在重合极限下是正则的，但依赖流形的整体性质，将它从物理发散中消去对重整化方案至关重要。在我们的情形中，$\bar{G}$ 仅出现在爱因斯坦张量的乘积项中，因此可以通过取背景几何在壳 (极限 $\varepsilon \to 0$) 下) 消去。或者，如 [1, 3] 所示，可以通过合适的波函数重整化 Z_h 消除非局域两圈发散，其中 $O\left(\varepsilon^{-1}\right)$ 为极点。该情形下，Z_h 的参数化要求涨落 $h_{\mu\upsilon}$ 中包含非线性项。共形规范下不带宇宙学常数的引力的最终两圈发散为:

$$\Gamma_\infty = \frac{G}{(4\pi)^2}\int \mathrm{d}^d x\sqrt{\bar{g}}\frac{3c-79}{8\varepsilon}\bar{R}. \tag{86}$$

The Renormalization of the Cosmological Constant

宇宙学常数的重整化

One can include the cosmological constant as a perturbation to the theory and renormalize it as a composite operator. We have:

我们可以将宇宙学常数作为理论的微扰引入，并将其作为复合算子进行重整化。我们得到:

$$\Lambda\int \mathrm{d}^d x\sqrt{g} = \Lambda\int \mathrm{d}^d x\sqrt{\bar{g}}\left(1+\frac{2\varepsilon b}{a}\psi\right)^{\frac{2d}{\varepsilon}} \tag{87}$$

$$\simeq \Lambda\int \mathrm{d}^d x\sqrt{\bar{g}}\exp\left\{\left(1-\frac{\varepsilon}{2}\right)\frac{\psi}{a}-\frac{\varepsilon}{8a^2}\psi^2+\ldots\right\}.$$

Keeping the quadratic order in ψ and computing the diagrams with one insertion of the composite operator, the only diagrams that are relevant are those with ψ propagators. One finds for the divergent parts of the effective action the following extra poles at one and two loops:

保留 ψ 的二次阶，计算插入一次该复合算子的图后，仅存的相关图是含 ψ 传播子的图。我们可得到有效作用量发散部分在单圈和双圈处的额外极点如下:

$$\Gamma^1_\Lambda = -\int \mathrm{d}^d x\sqrt{\bar{g}}\frac{G\Lambda}{4\pi}\left(\frac{1}{a\varepsilon}+\frac{1}{\varepsilon}\right)$$

$$\Gamma^2_\Lambda = \int \mathrm{d}^d x\sqrt{\bar{g}}\left[-\frac{G^2\Lambda}{16\pi^2}\left(\frac{1}{4a^2\varepsilon}+\frac{3}{\varepsilon^2}+\frac{35}{8\varepsilon}\right)+\frac{G^2\Lambda}{4\pi\varepsilon}\bar{G}_{\mu\nu}{}^{\mu\nu}+\frac{G}{4\pi\varepsilon}a^2\overline{\nabla}^2\bar{G}_{\psi\psi}\right], \tag{88}$$

where $\bar{G}_{\psi\psi}$ contains nonlocalities due to the propagator of ψ . As we can see from the structure of the two-loop divergence, we have two nonlocal divergences and an ε^2 pole. Once more, the term proportional to $\bar{G}_{\mu\upsilon}{}^{\mu\upsilon}$ can be removed by employing a particular wavefunction renormalization for ψ with nonlinear terms in $h_{\mu\nu}$. The term proportional to $\bar{G}_{\psi\psi}$ is due to one-loop subdivergences of the ghost and graviton fields, contributing to the divergent part of the kinetic term for ψ . We can get rid of this by including the counterterm in the one-loop action:

其中 $\bar{G}_{\psi\psi}$ 因 ψ 的传播子存在非局域性。从双圈发散的结构可以看出，我们得到两个非局域发散和一个 ε^2 极点。与 $\bar{G}_{\mu\nu}{}^{\mu\nu}$ 成正比的项可以通过对 ψ 采用含 $h_{\mu\nu}$ 非线性项的特定波函数重整化消去。与 $\bar{G}_{\psi\psi}$ 成正比的项来源于鬼场和引力子场的单圈次发散，贡献给 ψ 动能项的发散部分。我们可以通过在单圈作用量中引入 counterterm(抵消项) 来消去它:

$$\frac{(d-1)\,a^2}{2\pi\varepsilon}\int \mathrm{d}^d x\sqrt{\bar{g}}\left[\hat{R}\left(1+a\psi+\varepsilon b\psi^2\right)-\frac{1}{2}\partial_\mu\psi\partial^\mu\psi\right]. \tag{89}$$

Finally, the ε^2 pole is removed exploiting one’s freedom to rescale the background metric by a factor:

最后，利用重标度背景度规的自由度可以消去 ε^2 极点:

$$Z_{bg}=1-\left(\frac{G}{4\pi\varepsilon}\right)^2. \tag{90}$$

As a consequence, we get a rescaling of the Einstein-Hilbert action according to

由此，我们得到爱因斯坦-希尔伯特作用量按如下方式重标度

$$S_{EH}\to Z_{bg}^{\frac{\varepsilon}{2}}\int \mathrm{d}^d x\sqrt{g}R \tag{91}$$

All these final manipulations do not add extra divergences to the purely gravitational part discussed in the previous section. The final divergence for two-loop quantum gravity in $d=2+\varepsilon$ dimension in the conformal gauge turns out to be

所有这些最终操作都不会给上一节讨论的纯引力部分引入额外发散。共形规范下 $d=2+\varepsilon$ 维双圈量子引力的最终发散结果为

$$\Gamma_\infty=-3\frac{G}{(4\pi)^2}\int \mathrm{d}^d x\sqrt{\bar{g}}\frac{25-c}{8\varepsilon}\bar{R}. \tag{92}$$

We notice that Eq. (92) vanishes for $c=25$, which corresponds to the critical string. From (79), (89), and (92), one can extract the beta function for Newton’s coupling:

我们注意到，当 $c=25$ 对应临界弦时，式 (92) 为零。利用 (79)、(89) 和 (92)，可以提取出牛顿耦合的 β 函数:

$$\beta_G=\varepsilon G-\frac{25-c}{24\pi}G^2-5\frac{25-c}{48\pi^2}G^3. \tag{93}$$

The simple poles appearing in the renormalization of the cosmological constant and proportional to $\frac{1}{a^2}$ are due to the singular vev for ψ^2 :

宇宙学常数重整化中出现的与 $\frac{1}{a^2}$ 成正比的单极点来源于 ψ^2 的奇异真空期望值:

$$\langle \psi^2 \rangle = \frac{G}{2\pi\varepsilon}\left(1 + \frac{G}{16\pi}\right). \tag{94}$$

These poles can become important close to the UV fixed point, where a vanishes with the renormalization flow. One can perform a resummation of such contributions to obtain the following anomalous dimension for the volume operator:

这些极点在紫外不动点附近会变得重要，在该处 a 随重整化流变为零。我们可以对这类贡献做重求和，得到体积算子的反常维数如下:

$$\gamma_\Lambda = 2 - \frac{G}{8\pi} + \frac{8\pi a^2}{G} - \frac{8\pi}{G}\sqrt{a^4 + \frac{Ga^2}{2\pi} + \frac{1}{2}\left(\frac{Ga}{4\pi}\right)^2}. \tag{95}$$

In the deep ultraviolet, one has the fixed point $G^* = \frac{24\pi\varepsilon}{25-c} + o\left(\varepsilon^2\right)$ and a vanishes; therefore we may write:

在深紫外区，存在不动点 $G^* = \frac{24\pi\varepsilon}{25-c} + o\left(\varepsilon^2\right)$ 且 a 为零，因此我们可以写出:

$$\gamma_\Lambda^* = 2\left(1 - \frac{G^*}{16\pi}\right). \tag{96}$$

However, the physical content lies in the scaling relation between G and Λ which, in the UV, is given by

然而，物理内容蕴含在 G 和 Λ 之间的标度关系中，该关系在紫外区的形式为

$$\left(\frac{1}{G} - \frac{1}{G^*}\right)^{\varepsilon + \frac{3\varepsilon}{25-c}} \sim \Lambda^\varepsilon \tag{97}$$

as opposed to the scaling at the IR Gaußian fixed point that reads

与之相对，红外高斯不动点处的标度关系为

$$\frac{1}{G^d} \sim \Lambda^\varepsilon \tag{98}$$

Revisiting Quantum Gravity in $2 + \varepsilon$

重探 $2 + \varepsilon$ 中的量子引力

We discuss the insights coming from a modified renormalization scheme with different metric parameterizations. The main idea is to treat separately the dimensionality of spacetime that counts the degrees of freedom of the metric from the dimensionality that is analytically continued to regulate the covariant Feynman diagrams.

我们将讨论不同度规参数化下修正重整化方案带来的新见解。核心思路是将统计度量自由度的时空维度，与为正则化协变费曼图进行解析延拓得到的维度分开处理。

The success of the renormalization program for two-dimensional quantum gravity in the conformal gauge raises several questions. First of all, one would expect the two formulations of quantum gravity, i.e., the one conformal gauge and the one based on the Einstein-Hilbert action studied by Jack and Jones, to contain the same physics on the ground that the two forms of the microscopic action can be seen as two different but equivalent classical frames. However, the one-loop beta functions seem to have very different phenomenology due to a different central charge value. One may be tempted to attribute such difference to different universality classes since one-loop beta functions are expected to be scheme independent; however, it is unclear why a simple frame transformation would cause such a dramatic change in the path integral. A possible speculation is that the parametrization of the fluctuations could be the cause of the different results, because the linear difference $h_{\mu\upsilon} = g_{\mu\upsilon} - \bar{g}_{\mu\upsilon}$ parametrizes an integration domain that differs substantially in the space of metrics (unless imposing nonlinear Ward identities for the splitting symmetry; see section "Appendix A: Background Field Method"), as opposed to the exponential parametrization that is used in the conformal gauge, which preserves the integration domain by construction, i.e., constraining the metric to have a fixed signature [26]. Moreover, the difference in higher loop computations suggests that a more careful analysis of the degrees of freedom must apply, keeping in mind also the importance of the path-integral measure.

共形规范下二维量子引力重整化方案的成功引出了若干问题。首先，我们本应预期两种量子引力表述——即共形规范表述，以及杰克和琼斯研究的基于爱因斯坦-希尔伯特作用量的表述——包含相同的物理内容，因为微观作用量的两种形式可以看作两种不同但等价的经典规范。然而，由于中心荷取值不同，一圈 β 函数表现出的物理规律截然不同。由于一圈 β 函数通常被认为是方案无关的，人们很容易将这种差异归因为不同普适类；但目前尚不清楚，为何一个简单的规范变换会给路径积分带来如此剧烈的变化。一个可能的推测是，涨落的参数化方式导致了结果的差异，因为线性差异 $h_{\mu\upsilon} = g_{\mu\upsilon} - \bar{g}_{\mu\upsilon}$ 参数化的积分区域在度规空间中与指数参数化的结果存在本质不同 (除非对分裂对称性施加非线性沃德恒等式，参见“附录 A: 背景场方法”一节)；而共形规范中使用的指数参数化从构造上就保留了积分区域，也就是约束度规具有固定符号 [26]。此外，多圈计算的差异表明，我们必须更仔细地分析自由度，同时也不能忘记路径积分测度的重要性。

Keeping in mind the original idea from Weinberg, recent works [20, 21, 30, 31, 60, 61, 68] revived the computations of gravity in $d = 2 + \varepsilon$ dimensions trying to pay special attention to the scheme dependence of the renormalization flow and the parametrization of the symmetry group. As mentioned in section "The Story of the Conformal Mode in $2 + \varepsilon$ Dimensions", the kinematic pole as $d \to 2$ in the tree-level propagator does not originate due to our regularization scheme. It seems natural, also in light of the conformal gauge results, to keep the kinematic pole $(d-2)^{-1}$ explicit, without a priori identifying it with the (inverse of the) ε variable of dimensional regularization. The basic idea behind such a choice is that d , whenever entering a tensorial structure, parametrizes the gauge symmetry of gravity, much like the parameter N does in $SU(N)$ Yang-Mills theories. However, at the same time, when one is computing Feynman diagrams, the integration measure will be fictitiously shifted away from its critical dimension $d_c = 2$ with the only purpose of making the diagrams convergent, and this causes divergence of the theory to isolate in the form of ε -poles. The two "instances" of the dimension of spacetime can be equated later, after the theory has been properly renormalized, in order to extract results in some specific dimension. Finally, if one hopes to find a true physical meaning in the renormalized coupling constants of the theory, it is of crucial importance to identify the gauge- and parametrization-independent counterterms.

遵循温伯格最初的思路，近期研究 [20, 21, 30, 31, 60, 61, 68] 重新开展了 $d = 2 + \varepsilon$ 维引力的计算，特别关注重整化流的方案依赖性和对称群的参数化问题。正如“$2 + \varepsilon$ 维中共形模的故事”一节所述，树级传播子中作为 $d \to 2$ 出现的运动学极点并非我们的正则化方案产生的。结合共形规范的结果，很自然的处理方式是保留运动学极点 $(d-2)^{-1}$ 的显式形式，不提前将其等同于维数正则化的 ε 变量 (的逆)。这种选择背后的基本思路是，d 进入张量结构时，参数化了引力的规范对称性，就像参数 N 在 $SU(N)$ 杨-米尔斯理论中所做的那样。但与此同时，在计算费曼图时，为了让图收敛，积分测度会被人为地从临界维度 $d_c = 2$ 偏移，这会导致理论的发散以 ε 极点的形式分离出来。在理论完成适当重整化后，我们可以再将时空维度的两个“实例”等同起来，从而得到特定维度下的结果。最后，如果我们希望从理论重整化后的耦合常数中得到真正的物理意义，识别出规范和参数化无关的抵消项至关重要。

Computing the one-loop effective action using the regularization scheme described above from the action (62), one can inspect the dependence on the background splitting of the metric parametrizing it as

利用上述正则化方案，从作用量 (62) 计算一圈有效作用量，我们可以检验其对背景分裂的依赖，并将度规参数化为:

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} + \frac{\lambda}{2} h_{\mu\rho} \bar{g}^{\rho\theta} h_{\theta\nu}, \tag{99}$$

where we truncate the splitting to the quadratic order for simplicity, but further powers can in principle be included. Note that the symbol $\bar{g}$ now represents a different background with respect to the previous section, because here we do not isolate the conformal mode. The dependence on the gauge choice can be made explicit by including a deformation of the Feynman-de Donder gauge fixing (64):

其中为简化计算，我们将分裂截断到二阶，但原则上可以包含更高次幂。请注意，符号 $\bar{g}$ 在这里代表与上一节不同的背景，因为我们这里没有分离出共形模。通过引入费曼-德东德规范固定 (64) 的形变，可以将规范选择的依赖关系显式写出:

$$S_{gf}[h; \bar{g}] = \frac{1}{2} \int \mathrm{d}^d x \sqrt{\bar{g}} \bar{g}^{\mu\nu} F_\mu F_\nu, \tag{100}$$

$$F_\mu = \overline{\nabla}_\alpha h^\alpha_\mu - \frac{1 + \delta\beta}{2} \overline{\nabla}_\mu h^\alpha_\alpha.$$

The deformation parameter $\delta\beta$ is chosen such that $|\delta\beta| \ll 1$, also for simplicity.

形变参数 $\delta\beta$ 按照 $|\delta\beta| \ll 1$ 选取，同样是为了简化计算。

To elaborate on a subtle point: in a perturbative approach one could think of $\delta\beta$ as a new coupling and ”renormalize” the gauge fixing deformation as a composite operator, because it enters as an operator insertion in the loops when considering the leading order in $\delta\beta$. In practice, the independence of the final physical results from $\delta\beta$ tests that their derivative with respect to the gauge parameter is zero. It does not test that they are fully gauge-independent, although it provides a strong indication. In the following we consider only the leading order correction due to $\delta\beta$.

我们来详细阐述一个微妙的点: 在微扰方法中，我们可以将 $\delta\beta$ 视为一个新的耦合，并将规范固定变形作为复合算子进行“重整化”，因为在考虑 $\delta\beta$ 的领头阶时，它是以算子插入的形式进入圈图的。实际上，最终物理结果不依赖于 $\delta\beta$ ，这只能验证它们对规范参数的导数为零。这并不能证明它们是完全规范无关的，只是提供了一个强有力的迹象。在下文中，我们仅考虑 $\delta\beta$ 带来的领头阶修正。

As before, Feynmann integrals are formally computed below $d = 2$, but we report the result continued above two dimension. We get

和之前一样，费曼积分在 $d = 2$ 以下进行形式计算，但我们给出解析延拓到二维以上的结果。我们得到

$$\Gamma_\infty = -\frac{\mu^\varepsilon}{\varepsilon}\int \mathrm{d}^d x\sqrt{\bar{g}}\left[A\bar{R} + J_{\mu\nu}\left(G^{\mu\nu} + \frac{1}{2}\lambda G\bar{g}^{\mu\nu}\right)\right], \tag{101}$$

where $G^{\mu\nu} = \bar{R}^{\mu\nu} - \frac{1}{2}\bar{R}\bar{g}^{\mu\nu}$ is Einstein’s tensor for the background metric [60]. As for the coefficients, we have:

其中 $G^{\mu\nu} = \bar{R}^{\mu\nu} - \frac{1}{2}\bar{R}\bar{g}^{\mu\nu}$ 是背景度规的爱因斯坦张量 [60]。系数满足如下关系:

(102)

$$A = \frac{36 + 3d - d^2}{48\pi}$$

$$J_{\mu\nu} = \frac{\bar{g}_{\mu\nu}}{4\pi}\left[\frac{d^2 - d - 4}{2(d-2)}\lambda - \delta\beta\left(2 + \frac{2\lambda}{d-2}\right) - d - 1\right],$$

We notice, as expected, that going on-shell eliminates both the parametrization and gauge dependences, because $J_{\mu\nu}$ decouples. Moreover, the possible extra poles of kinematical origin appear only in combination with the parameter λ and also disappear on-shell.

我们注意到，不出所料，取在壳条件会同时消除参数化依赖和规范依赖，因为 $J_{\mu\nu}$ 退耦了。此外，运动学来源的可能额外极点仅与参数 λ 结合出现，在壳条件下也会消失。

The role of $J_{\mu\nu}$ is that of a wavefunction renormalization that needs to be included in higher loop computations. At subleading order in the renormalization with this scheme, which are not available yet, the source $J_{\mu\nu}$ must be used to dress external graviton lines, and thus produces a notion of ”quantum metric,” which is the natural argument of the effective action that is also fully dressed by quantum corrections. Strictly speaking, since the metric is not an observable, it is not strange that its renormalization carries the parametric and gauge dependencies. Although this does not prove renormalizability of Einstein-Hilbert formulation in $d = 2 + \varepsilon$, it suggests that higher loop computation might turn out to be less pathological than those expected in Ref. [53].

$J_{\mu\nu}$ 的作用是波函数重整化，需要包含在高阶圈图计算中。在该方案的次领头阶重整化 (目前尚无相关结果) 中，必须用源 $J_{\mu\nu}$ 修饰外引力子线，由此产生了“量子度规”的概念；量子度规是有效作用量的自然宗量，本身也被量子修正完全修饰。严格来说，既然度规不是可观测量，它的重整化带有参数依赖和规范依赖并不奇怪。尽管这不能证明爱因斯坦-希尔伯特表述在 $d = 2 + \varepsilon$ 下可重整，但它表明高阶圈图计算或许不会像文献 [53] 预期的那样存在大量问题。

It is important to notice how the beta function for Newton's constant

需要重点注意牛顿常数的 β 函数

$$\beta_G = \varepsilon G - \frac{36 + 3d - d^2}{48\pi} G^2, \tag{103}$$

is parametrically finite in any d and gauge-independent, with a UV interacting fixed point of order ε that disappears in dimension $d_c \approx 7.7$ [30, 54, 60] , which contains the physical case of four dimensions. However, the result is perturbative, and it is hard to expect that its predictions could be quantitatively accurate. Despite the limitations, this is one important result in that it finds that the gravitational fixed point exists within a conformal window, which is different from the naive prediction of the functional renormalization group in the background field [56]. In the limit $d \to 2$ the beta function (103) reproduces the "central charge" value, $c = -19$ of Ref. [53].

在任意 d 下都是参数有限且规范无关的，它存在一个阶为 ε 的紫外相互作用不动点，该不动点在维度 $d_c \approx 7.7$ [30, 54, 60] (包含四维这一物理情形) 中消失。但该结果是微扰的，很难期望它的预言能在定量上准确。尽管存在局限性，这仍是一项重要结果: 它发现引力不动点存在于共形窗口内，这与背景场下泛函重整化群的朴素预测不同 [56]。在 $d \to 2$ 极限下，β 函数 (103) 重现了文献 [53] 给出的“中心荷”值 $c = -19$ 。

One can use the same prescription to investigate the formulation of gravity invariant under Diff*. As we pointed out in section "The Story of the Conformal Mode in $2 + \varepsilon$ Dimensions", the best way to parametrize fluctuations of the metric is through formula (74), so that the background metric lies in the same unimodular class as the quantum metric. In this case one has multiple choices to impose the on-shell condition. One natural choice is to use the background equations of motion for the dilaton field, denoted e in the following, and parametrize the divergent part of the theory. Starting from the background action (without cosmological constant)

我们可以用相同的规定研究在 Diff* 下不变的引力表述。正如我们在“$2 + \varepsilon$ 维中共形模式的来龙去脉”一节指出的，对度规涨落参数化的最佳方式是公式 (74)，这样背景度规与量子度规属于同一个幺模类。在这种情况下，我们有多种施加在壳条件的选择。一种自然的选择是利用膨胀子场的背景运动方程 (下文中记为 e)，对理论的发散部分进行参数化。从背景作用量 (无宇宙学常数) 出发

$$S[\bar{g}, \psi] = -\frac{1}{G} \int \mathrm{d}^d x \sqrt{\bar{g}} \left[\psi^2 \bar{R} + 4 \frac{d-1}{\varepsilon} \bar{g}^{\mu\nu} \partial_\mu \psi \partial_\nu \psi \right] \tag{104}$$

with $\bar{g}$ a unimodular metric, the equations of motion read:

其中 $\bar{g}$ 是幺模度规，运动方程为:

$$e \equiv 8 \frac{d-1}{\varepsilon G} \overline{\nabla}^2 \psi - \frac{2}{G} \bar{R} \psi = 0. \tag{105}$$

Again, we include a deformation of the gauge fixing (76) that now reads

我们再次引入规范固定 (76) 的变形，此时变形后的规范固定为

$$S_{gf}\left[h,\psi;\bar{g}\right]=\frac{1}{2}\int \mathrm{d}^{d}x\sqrt{\bar{g}}\bar{g}^{\mu\upsilon}F_{\mu}F_{\upsilon}, \tag{106}$$

$$F_{\mu}=\psi_{0}\overline{\nabla}^{\alpha}h_{\alpha\mu}-2\left(1+\delta\beta\right)\partial_{\mu}\psi,$$

where ψ_0 is the (nonconstant) background for the conformal mode (Note the different normalization of ψ already evident comparing (104) and (72).)

其中 ψ_0 是共形模式的 (非恒定) 背景 (注意比较 (104) 和 (72) 即可发现 ψ 的归一化不同)。

The divergent part of the effective action in the background field method is then given by

背景场方法中有效作用量的发散部分由下式给出

$$\Gamma_{\infty}=-\frac{\mu^{\varepsilon}}{\varepsilon}\int \mathrm{d}^{d}x\sqrt{\bar{g}}\left[B\psi_{0}^{2}\bar{R}+J\psi_{0}e\right], \tag{107}$$

where $\bar{R}$ is the Ricci scalar for the unimodular background. The coefficients B and J are given by

其中 $\bar{R}$ 是幺模背景的里奇标量。系数 B 和 J 由下式给出

$$B=-\frac{11d^{4}-44d^{3}-78d^{2}+180d-72}{96\pi d\left(d-1\right)}$$

$$-\frac{\left(d-2\right)\left(18d^{5}-35d^{4}-132d^{3}+152d^{2}+48d-48\right)}{192\pi d^{3}\left(d-1\right)}\delta\beta, \tag{108}$$

$$J=-\frac{3d^{3}-6d^{2}-12d+16}{8\pi d}-\frac{\left(d-2\right)\left(3d^{2}-4\right)\left(2d^{2}+d-2\right)}{16\pi d^{3}}\delta\beta.$$

The first thing to notice is that a residual gauge dependence for the on-shell counterterm survives if $d\neq 2$. This is not surprising since the Weyl symmetry used to construct the gauge group is not even realized at tree level for arbitrary dimension. In the limit $d\to 2$ the residual gauge dependence vanishes on-shell, and one recovers the beta function:

需要注意的第一点是，若 $d\neq 2$ ，壳上抵消项仍会残留规范依赖性。这并不奇怪，因为用于构造规范群的外尔对称性，在任意维度的树图阶都无法实现。当 $d\to 2$ 时，残留规范依赖性在壳上消失，即可得到 β 函数:

$$\beta_{G}=-\frac{25}{24\pi}G^{2}. \tag{109}$$

However, while (103) was obtained by fixing Λ and measuring the flow of G with respect to the volume operator, here we have not yet considered the cosmological constant. In fact, the use of the conformal mode to impose the on-shell condition boils down to a different scheme and a different choice of essential couplings. To have a proper comparison, one can study the renormalization of the volume operator as a composite of the dilaton field given by

但需要注意，(103) 式是通过固定 Λ 、测量 G 相对于体积算符的流得到的，而本文尚未考虑宇宙学常数。实际上，利用共形模施加壳条件本质上对应不同的方案和不同的基本耦合选择。若要进行恰当对比，可以研究由 dilaton 场复合而成的体积算符的重整化，形式为

$$V[\psi] = \Lambda \int \mathrm{d}^d x \sqrt{\bar{g}} \psi^{\frac{2d}{\varepsilon}}, \tag{110}$$

and impose the relation

并施加如下关系

$$\Lambda = \frac{\psi^{-\frac{d+2}{d-2}}}{dG} \left[(d-2) \bar{R} \psi - 4(d-1) \overline{\nabla}^2 \psi \right]. \tag{111}$$

In so doing one recovers, in the limit $d \to 2$, the beta function (103) for the Newton's constant. This reveals that the two results for the values of the central charge are actually due to a difference in the scheme used to go on-shell. Consequently, the value $c = 25$ comes from a parametrization of the group of diffeomorphisms that contains the Weyl group and from preserving the subgroup generated by Weyl transformations in the use of the equations of motions, but this is possible only in $d = 2$.

通过这种方式，在 $d \to 2$ 极限下可以重新得到牛顿常数的 β 函数 (103)。这表明，中心荷取值的两个不同结果实际上源于进入壳时采用的方案不同。因此，$c = 25$ 的取值来自对包含外尔群的微分同胚群的参数化，以及在运动方程中保留外尔变换生成的子群，但这仅在 $d = 2$ 中可以实现。

As a final remark, it is interesting to notice that, in $d = 2$, one way to realize a Weyl-symmetric quantum theory is to first break explicitly the symmetry at tree level with a term of the form

最后需要指出一个有意思的结论: 在 $d = 2$ 中，实现外尔对称量子理论的一种方式是，先通过一项如下形式的项在树图阶显式破缺该对称性

$$\Delta S_t [g, \psi] = q \int \mathrm{d}^2 x \sqrt{g} R \psi. \tag{112}$$

The running of the coupling q , known occasionally as topological charge, under the renormalization flow is trivial (only in $d = 2$); therefore, one can set it to any value that cancels the one-loop Weyl anomaly induced by the other terms. However, using the prescription explained above, one can explicitly check that q actually renormalizes for $d > 2$ and so it is not useful to restore the invariance. Again, this is compatible with the fact that the symmetry is well defined only in $d = 2$.

耦合 q (有时被称为拓扑荷) 在重整化流下的跑动是平凡的 (仅在 $d = 2$ 中成立)；因此我们可以将其设为任意值，抵消其他项诱导的单圈外尔反常。但使用上文介绍的规则可以明确验证，对于 $d > 2$ ，q 实际上会发生重整化，因此无法用于恢复不变性。这再次符合该对称性仅在 $d = 2$ 中良定义的结论。

All these computations show that, although unimodular dilaton gravity is well-suited for applications to two-dimensional gravity and string theory (where the anomaly can be canceled through the topological charge), most of its nice properties do not hold beyond $d = 2$ and is not a good candidate to be analytically continued until $d=4$ as suggested by Weinberg. The natural candidate is instead the formulation based on the Diff symmetry group with renormalization discussed in Eq. (101).

所有这些计算表明，尽管幺模 dilaton 引力非常适合应用于二维引力和弦论 (这类场景中反常可以通过拓扑荷抵消)，但它大部分优良性质在超出 $d=2$ 后都不再成立，也并不像温伯格建议的那样适合解析延拓到 $d=4$ 。反之，最合适的候选方案是基于 Diff 对称群、并经过 (101) 式讨论的重整化的表述。

Conclusions

结论

It is difficult to forecast the future of Weinberg's and Reuter's asymptotic safety program for quantum gravity. While we know very well that new computations - increasingly technical and complex and, most likely, based on Wetterich's incarnation of the functional renormalization group method - will probably further confirm the original conjecture by finding an appropriate ultraviolet fixed point, we must acknowledge that eventual scheme, gauge, or parametric dependences of the results will not convince presently skeptical fellow scientists to endorse the program.

我们很难预测温伯格和罗伊特提出的量子引力渐近安全方案的未来。我们清楚，越来越技术化、复杂化，且极有可能基于韦特里奇版本泛函重整化群方法的新计算，大概率会通过找到合适的紫外不动点进一步证实最初的猜想，但我们必须承认，若结果最终存在方案、规范或参数依赖性，就无法说服目前持怀疑态度的同行认可这个方案。

In this chapter, we tried to give a path, in part logical and in part historical, which describes the onset of the original idea by Weinberg and its developments. We covered mostly the early years, but also connected with recent research topics. We purposedly skewed our narrative toward the perturbative results, which can explicitly be shown to be independent of all unwanted parameters, including the gauge ones, at least at the leading order in the expansion in $d=2+\varepsilon$ dimensions.

在本章中，我们尝试从逻辑与历史两个维度梳理脉络，介绍温伯格最初构想的提出与后续发展。我们主要涵盖了研究早期的内容，同时也关联了近年的研究课题。我们有意将叙述偏向微扰结果，至少在 $d=2+\varepsilon$ 维展开的领头阶下，可以明确证明这类结果不依赖所有不必要参数，包括规范参数。

Our hope is that a renewed interest toward the perturbative perspective in $d=\ 2+\varepsilon$ will go hand in hand with the future nonperturbative developments in $d=4$. In fact, each approach has something to teach us and both can learn from each other. We could outline two important lessons coming from the perturbative approach (among the many) that should be considered on the nonperturbative side.

我们希望，针对 $d=\ 2+\varepsilon$ 下微扰视角的重新关注，能和未来 $d=4$ 中的非微扰发展齐头并进。事实上，两种方法各有可取之处，还能相互借鉴。我们可以归纳出微扰方法给出的两个重要经验，值得非微扰研究参考。

The first, although trivial, lesson is that it is necessary to go on-shell in order to show the gauge independence of the final results, which is a well-known fact of quantum field theory. This is a step that is, however, not as easy to take in the current applications of the functional renormalization group approach, considering also that few attempts have ever been made (see, e.g., Refs. [7, 8, 10]). This step could have notable consequences when determining, for example, the dimensionality of the ultraviolet critical surface (i.e., the predictive power of the theory) in terms of the number of scaling operators that actually exist on-shell and not only off-shell.

> 第一个经验虽微不足道，但十分必要: 要证明最终结果不依赖规范，就必须采用在壳方案，这是量子场论中广为人知的结论。但在泛函重整化群方法的当前应用中，这一步并不容易实现，目前也极少有尝试 (参见例如文献 [7, 8, 10])。这一步会带来显著影响，例如在根据实际存在的在壳 (而非仅离壳) 标度算符数量确定紫外临界曲面的维数 (即理论的预言能力) 时，就会体现出来。

A second lesson comes from the fact that dimensional regularization requires that the dimensionality is continued below $d = 2$ for Feynman diagrams to be finite, but the results must be continued above $d = 2$ in order to apply to the physical four-dimensional case. This is strictly related to the problem of the (perturbative) instability of the conformal mode, and we recommend the general discussion of Ref. [55] for some insights and implications. It is known from simpler field theories (see, e.g., Ref. [41]) that this type of continuation can result in strong instanton contributions that cannot be neglected in the nonperturbative definition of the path-integral, and has the potential side effect of making the theory non-unitary. In fact, unitarity itself is an often forgotten issue of the asymptotic safety proposal, although some work has been done in that direction [5,68].

> 第二个经验来自维度正规化的要求: 为了让费曼图有限，需要将维数解析延拓到 $d = 2$ 以下，但要得到物理四维情形的结果，又必须将结果延拓回 $d = 2$ 以上。这和共形模的 (微扰) 不稳定性问题直接相关，我们推荐阅读文献 [55] 的总体讨论，其中有相关见解与推论。从更简单的场论中我们已经得知 (参见例如文献 [41])，这类延拓会产生不可忽略的瞬子贡献，影响路径积分的非微扰定义，还可能产生副作用，导致理论失去幺正性。实际上，幺正性本身就是渐近安全方案中经常被忽略的问题，尽管已经有部分相关研究 [5,68]。

Important toy models that have the potential to teach us how asymptotically safe quantum gravity should behave above $d = 2$ are $SU(N)$ Yang-Mills theories above $d = 4$ (see [51] and an astute footnote of Ref. [72]). In fact, Yang-Mills gauge theories are asymptotically free in $d = 4$ for some values of the parameters (like gravity in $d = 2$), so they are expected to be asymptotically safe above $d = 4$ (like gravity is supposed to be above $d = 2$). The case of $SU(2)$ Yang-Mills in five dimensions has been studied in the past and, recently, both on the lattice [33, 50, 93], where there is evidence only of a first-order phase transition (which in layman terms means bad luck: no ultraviolet fixed point!) and with the resummation of the perturbative series [25, 63, 93] , where error bars caused by nonperturbative effects become quite considerable in $d = 5$. Conversely, the application of the functional renormalization group with the background field method undoubtedly suggests that there is a nontrivial ultraviolet completion [40]. In short, regarding especially the change with the dimensionality of the nature of the UV behavior, the only evidence for local gauge theories that become asymptotically safe (above the dimensionality in which they are asymptotically free) is based on functional renormalization group methods (This is not in contradiction with the existence of asymptotically safe gauge Yukawa theories in four dimensions.). This is certainly a point that deserves a careful consideration in the future.

有一类重要的玩具模型，能够帮助我们理解渐近安全量子引力在 $d=2$ 以上的行为，这就是 $SU(N)$ 中高于 $d=4$ 的杨-米尔斯理论 (见文献 [51] 及参考文献 [72] 中一条敏锐的脚注)。实际上，杨-米尔斯规范理论在 $d=4$ 下对部分参数取值是渐近自由的——就像引力在 $d=2$ 下的情况，因此人们预期它们在 $d=4$ 以上是渐近安全的——正如引力本应在 $d=2$ 以上是渐近安全的。五维 $SU(2)$ 杨-米尔斯理论的情况过去已有研究，近来人们同时从格点量子场论 [33, 50, 93] 和微扰级数重求和 [25, 63, 93] 两个方向展开工作: 格点研究仅找到一级相变的证据 (通俗来说这意味着不走运: 没有紫外不动点！)；而在微扰级数重求和中，非微扰效应带来的误差棒在 $d=5$ 下会变得相当大。与之相反，结合背景场方法的泛函重整化群应用明确表明该理论存在非平庸的紫外完备性 [40]。简而言之，尤其就紫外行为的性质随维度的变化而言，对于局域规范理论在其渐近自由的维度之上变为渐近安全的情况，目前仅有的证据来自泛函重整化群方法 (这和四维中存在渐近安全规范汤川理论并不矛盾)。这一点显然值得未来仔细研究。

As a final remark, we point out the importance of the definition of the path integral. Several elements have to be considered when searching for the proper quantum theory of gravity, ultimately corresponding to a choice of functional measure in the partition function. Two elements that often reflect such (implicit) choice are the symmetry group and the parametrization of the fluctuations. In the case of quantum gravity and asymptotic safety, a fundamental question concerns the role of Weyl symmetry and, potentially, its anomaly. However, we realized how requiring the theory to be defined by an analytic continuation in the dimension d constrains some of these aspects, mostly because of the different realizations of conformal (classical) theories in specific dimensions. Of course, a different approach as the functional renormalization group might restrict the analysis to a fixed dimensionality and give an entirely different point of view on the four-dimensional conformal anomaly.

最后我们需要指出，路径积分的定义十分重要。在寻找正确的引力量子理论时，必须考虑多个要素，这些要素最终对应配分函数中泛函测度的选择。对称性群和涨落参数化是两个常反映这种 (隐含) 选择的要素。在量子引力和渐近安全的研究中，一个核心问题关乎外尔对称性的作用，以及可能存在的外尔反常。但我们已经认识到，要求理论通过维度 d 的解析延拓定义，会对其中部分方面做出限制，这主要是因为共形 (经典) 理论在特定维度有不同的实现形式。当然，像泛函重整化群这类不同的方法，或许可以将分析限制在固定维度，为四维共形反常提供完全不同的视角。

Appendix A: Background Field Method

附录 A: 背景场方法

We present a brief and informal outline of the background field method that summarizes the most important fetures of its application and the motivations behind it. The standard path-integral of an (Euclidean) field theory is based on the source-dependent functionals

我们对背景场方法给出一份简略的非正式概述，总结其应用最重要的特点以及其背后的研究动机。(欧氏) 场论的标准路径积分基于依赖源的泛函

$$Z[J] = \mathrm{e}^{W[J]} = \int D\phi \mathrm{e}^{-S[\phi]+J\cdot\phi}, \tag{113}$$

where J is the source for the field ϕ and the internal product consists in a sum over all internal indices and a covariant integration over spacetime. The effective action is a functional of $\overline{\phi} = \langle\phi\rangle_J$, compactly defined as

其中 J 是场 ϕ 的源，内积包括对所有内指标求和以及对时空的协变积分。有效作用量是 $\overline{\phi} = \langle\phi\rangle_J$ 的泛函，可简洁定义为

$$\mathrm{e}^{-\Gamma[\overline{\phi}]} = \int D\phi \mathrm{e}^{-S[\phi]+\frac{\delta\Gamma}{\delta\overline{\phi}}\cdot(\phi-\overline{\phi})}, \ \overline{\phi} = \frac{\delta W}{\delta J}. \tag{114}$$

In the background approach, one follows the strategy of decomposing the field ϕ into a background φ and fluctuations χ , with the aim of integrating over χ . The simplest choice is the linear split, $\phi = \varphi + \chi$, though nonlinear implementations might be useful in some circumstance. The background version of (113) is

在背景场方法中，我们采用将场 ϕ 分解为背景场 φ 和涨落场 χ 的策略，目标是对 χ 做积分。最简单的选择是线性分解，即 $\phi = \varphi + \chi$ ，不过非线性分解在某些情况下也很有用。式 (113) 的背景场形式为

$$Z_\varphi[J] = \mathrm{e}^{W_\varphi[J]} = \int D\chi \mathrm{e}^{-S[\varphi+\chi]+J\cdot\chi}, \tag{115}$$

where the dependence on the background φ is kept parametrically. The background effective action is

其中对背景 φ 的依赖以参数形式保留。背景场有效作用量为

$$\mathrm{e}^{-\Gamma_\varphi[\overline{\chi}]} = \int D\chi \mathrm{e}^{-S[\varphi+\chi]+\frac{\delta\Gamma_\varphi}{\delta\overline{\chi}}\cdot(\chi-\overline{\chi})}, \ \overline{\chi} = \frac{\delta W_\varphi}{\delta J}. \tag{116}$$

There are two intimately connected reasons to apply the background field method. On the one hand, we can consider the "zero-point function," i.e., $\overline{\chi} = 0$, and obtain a functional $\overline{\Gamma}[\varphi] \equiv \Gamma_\varphi\left[\overline{\chi} = 0\right]$ that depends on a single field, which we argue briefly is (related to) the traditional effective action $\Gamma\left[\overline{\phi}\right]$. On the other hand, the background field method allows us to preserve any symmetry, including nonlinear and gauge ones, throughout the computation in some form.

应用背景场方法有两个密切相关的原因。一方面，我们可以研究“零点函数”，即 $\overline{\chi} = 0$ ，得到仅依赖单个场的泛函 $\overline{\Gamma}[\varphi] \equiv \Gamma_\varphi\left[\overline{\chi} = 0\right]$ ，我们会简要说明它就是 (关联于) 传统有效作用量 $\Gamma\left[\overline{\phi}\right]$ 。另一方面，背景场方法能让我们在整个计算过程中以某种形式保留所有对称性，包括非线性对称性和规范对称性。

To argue the first point, consider the "split" symmetry induced by the arbitrary split $\phi = \varphi + \chi$:

为了说明第一点，我们考虑由任意分解 $\phi = \varphi + \chi$ 诱导的“分解对称性”：

$$\varphi \to \varphi - B, \ \chi \to \chi + B. \tag{117}$$

By construction $S[\varphi+\chi]$ is invariant under this transformation. Formally, the original action satisfies the Nöther identity $\frac{\delta S}{\delta\varphi}=\frac{\delta S}{\delta\chi}$ implying that it is a function of the sum, $\phi=\varphi+\chi$. Using (116), one derives the split Ward identities:

根据构造，$S[\varphi+\chi]$ 在该变换下不变。形式上，原作用量满足诺特恒等式 $\frac{\delta S}{\delta\varphi}=\frac{\delta S}{\delta\chi}$ ，这意味着它是求和 $\phi=\varphi+\chi$ 的函数。利用式 (116)，可以推导出分解沃德恒等式:

$$\frac{\delta\Gamma_\varphi}{\delta\varphi}=\frac{\delta\Gamma_\varphi}{\delta\overline{\chi}}+\mathcal{A}, \tag{118}$$

where $\mathcal{A}\propto\langle\delta\log D\chi\rangle$ is some eventual anomaly induced by the noninvariance of $D\chi$ under (117) (further contributions are present for nonlinear splits generalizing (117)). Dimensional regularization is a procedure such that oftentimes $\mathcal{A}=0$, so $\Gamma_\varphi\left[\overline{\chi}\right]$ must be a function of the sum, which can be denoted $\overline{\phi}=\varphi+\overline{\chi}$. As a consequence the (zero-point function of the) background effective action and the original effective action are the same:

其中 $\mathcal{A}\propto\langle\delta\log D\chi\rangle$ 是 $D\chi$ 在变换 (117) 下因非不变性产生的可能反常，(对于推广 (117) 的非线性分解，还存在额外贡献)。维数正规化通常会使得 $\mathcal{A}=0$ ，因此 $\Gamma_\varphi\left[\overline{\chi}\right]$ 必定是该求和的函数，可以记为 $\overline{\phi}=\varphi+\overline{\chi}$ 。因此，背景有效作用量 (的零点函数) 与原有效作用量完全一致:

$$\overline{\Gamma}\left[\overline{\phi}\right]=\Gamma\left[\overline{\phi}\right]. \tag{119}$$

As for the second point concerning the preservation of symmetries, suppose that $\phi\to\phi'=F[\phi]$ is nonanomalous gauge and/or nonlinear symmetry of the action, $S[\phi]=S[\phi']$. We deduce that it must be a symmetry of the effective action of (114); however, using standard Feynman diagrams and perturbation theory, it might not be obvious how to make it manifest at any stage of the computation. In the background path integral (116), the field is split, so there are two ”natural” realizations of the symmetry. The background symmetry, in which the background changes as would the original field, while the fluctuation χ transforms linearly (in χ itself). The full (quantum) symmetry, in which the background remains invariant and the fluctuation takes over the full transformation originally belonging to ϕ (i.e., generally, nonlinear in the fluctuation itself).

关于第二点对称性保留，假设 $\phi\to\phi'=F[\phi]$ 是作用量 $S[\phi]=S[\phi']$ 的无反常规范对称性和/或非线性对称性。我们可以推导出它必然是式 (114) 中有效作用量的对称性；但使用标准费曼图和微扰论时，在计算的任意阶段都很难让该对称性明显成立。在背景场路径积分 (116) 中，场经过分解后，对称性有两种“自然的”实现方式。一种是背景对称性: 背景场按原场的变换规则变换，而涨落场 χ 做线性变换 (相对于涨落场自身而言)。另一种是全 (量子) 对称性: 背景场保持不变，涨落场承担原本属于 ϕ 的全部变换 (即一般来说，变换相对于涨落场自身是非线性的)。

The background symmetry can be realized manifestly when covariantly expanding the action of any field (some attention must be given to nonlinear symmetries, but it is otherwise simple and straightforward). The full quantum symmetry, on the other hand, is the physically relevant symmetry (e.g., the one that has to be gauge-fixed), because the background field is purely a computational device and should not transform. However, the split Ward identities (118), if satisfied and nonanomalous, allow to change the relative contributions of background and fluctuations. As a consequence, it is simple to deduce that, if the background and split symmetries are satisfied, then the full quantum symmetry must also be satisfied!

对任意场的作用量进行协变展开时，背景对称性可以显式实现 (非线性对称性需要额外注意，除此之外过程简单直接)。另一方面，完整量子对称性才是具有物理相关性的对称性 (例如是必须进行规范固定的对称性)，因为背景场纯粹是计算工具，不应当发生变换。不过，若分裂沃德恒等式 (118) 成立且无反常，就可以调整背景和涨落的相对贡献。由此可以直接推导出: 若背景对称性和分裂对称性成立，那么完整量子对称性也必然成立!

Background Field: The Case of Metric Gravity

背景场: 度量引力的情形

In theories with a dynamical metric $g_{\mu\nu}$, the relevant symmetry to consider is diffeomorphism invariance. Infinitesimally, a diffeomorphism is parametrized by a tangent vector field ξ^μ and the metric transforms as $\delta_\xi g_{\mu\upsilon} = 2\nabla_{(\mu}\xi_{\upsilon)}$, where the index of ξ^μ is lowered by the metric $g_{\mu\nu}$. If we perform a simple split over a background $\bar{g}_{\mu\upsilon}$, we have that $g_{\mu\upsilon} = \bar{g}_{\mu\upsilon} + h_{\mu\upsilon}$, where the fluctuations $h_{\mu\upsilon}$ are integrated over in the path integral. With this procedure, we have the identifications $\phi \to g_{\mu\nu}$ and $\chi \to h_{\mu\nu}$ of the previous section; the effective action is thus expected to be a functional of $\overline{\chi} \to \langle h_{\mu\nu}\rangle$ that depends parametrically on the background, denoted $\Gamma_{\bar{g}}[\langle h\rangle]$. Notice that $\Gamma_{\bar{g}}[\langle h\rangle]$ can be understood as a "bimetric" functional $\Gamma[\langle g\rangle, \bar{g}] = \Gamma[\bar{g} + \langle h\rangle, \bar{g}]$, as was occasionally done in the asymptotic safety literature.

在带有动力学度量 $g_{\mu\nu}$ 的理论中，需要考虑的相关对称性是微分同胚不变性。无穷小微分同胚由切向量场 ξ^μ 参数化，度量按照 $\delta_\xi g_{\mu\upsilon} = 2\nabla_{(\mu}\xi_{\upsilon)}$ 变换，其中 ξ^μ 的指标由度量 $g_{\mu\nu}$ 降阶。如果我们对背景 $\bar{g}_{\mu\upsilon}$ 做简单分解，可得 $g_{\mu\upsilon} = \bar{g}_{\mu\upsilon} + h_{\mu\upsilon}$ ，其中涨落 $h_{\mu\upsilon}$ 会在路径积分中被积分掉。通过这一步骤，我们得到上一节给出的关系 $\phi \to g_{\mu\nu}$ 和 $\chi \to h_{\mu\nu}$ ；因此有效作用量可认为是 $\overline{\chi} \to \langle h_{\mu\nu}\rangle$ 的泛函，参数化依赖于背景，记为 $\Gamma_{\bar{g}}[\langle h\rangle]$ 。注意 $\Gamma_{\bar{g}}[\langle h\rangle]$ 可以理解为“双度量”泛函 $\Gamma[\langle g\rangle, \bar{g}] = \Gamma[\bar{g} + \langle h\rangle, \bar{g}]$ ，渐近安全文献中偶尔会采用这种理解。

The background transformations that are preserved through the quantization procedure are

量子化过程保留的背景变换为

$$\delta^b_\xi \bar{g}_{\mu\nu} = g_{\rho(\mu}\overline{\nabla}_{\nu)}\xi^\rho, \; \delta^b_\xi h_{\mu\nu} = \xi^\rho\overline{\nabla}_\rho h_{\mu\nu} + 2h_{\rho(\mu}\overline{\nabla}_{\nu)}\xi^\rho, \tag{120}$$

where we used the symmetric background connection $\overline{\nabla}_\rho \bar{g}_{\mu\nu} = 0$. The full quantum transformation is instead a complete diffeomorphism at fixed background, $\delta_\xi \bar{g}_{\mu\nu} = 0$, which is thus nonlinear in $h_{\mu\upsilon}$

其中我们使用了对称背景联络 $\overline{\nabla}_\rho \bar{g}_{\mu\nu} = 0$ 。而完整量子变换是固定背景下的完整微分同胚 $\delta_\xi \bar{g}_{\mu\nu} = 0$，因此它对 $h_{\mu\upsilon}$ 是非线性的

$$\delta_\xi h_{\mu\upsilon} = g_{\rho(\mu)}\nabla_\upsilon \xi^\rho = \bar{g}_{\rho(\mu}\overline{\nabla}_{\upsilon)}\xi^\rho + \xi^\rho\overline{\nabla}_\rho h_{\mu\nu} + 2h_{\rho(\mu}\overline{\nabla}_{\upsilon)}\xi^\rho + \mathcal{O}\left(h^2\right),$$

(121)

where $\mathcal{O}\left(h^2\right)$ hides computable, but more complicate, higher-order contributions. Nonlinear generalizations of the split do not affect the background transformation of the background metric, but they do add further nonlinearities to the transformations of $h_{\mu\nu}$. These, however, can also be computed iteratively without much difficulty.

其中 $\mathcal{O}(h^2)$ 包含了可计算但更为复杂的高阶贡献。分解的非线性推广不会改变背景度量的背景变换，但会给 $h_{\mu\nu}$ 的变换引入额外非线性。不过这些额外非线性也可以通过迭代轻松计算得到。

The advantage of the background field method, assuming that the split symmetry is not anomalous, is that computations of the effective action $\overline{\Gamma}[\bar{g}_{\mu\nu}] = \Gamma_{\bar{g}}[\langle h_{\mu\nu}\rangle = 0]$ are performed in the limit $\langle h_{\mu\nu}\rangle = 0$, so, at the leading order, transformations of $h_{\mu\upsilon}$ can be largely ignored (although they do become important at next-to-leading order and, in general, whenever higher vertices with external $h_{\mu\upsilon}$ lines are needed). It is straightforward to see that $\delta_\xi h_{\mu\upsilon}|_{h=0} = \delta^b_\xi \bar{g}_{\mu\upsilon}$, which is another way to see that the single-field effective $\overline{\Gamma}[\bar{g}_{\mu\upsilon}]$ is expected to be diffeomorphism invariant when split symmetry is not anomalous.

若分解对称性没有反常，背景场方法的优势在于: 有效作用量 $\overline{\Gamma}[\bar{g}_{\mu\nu}] = \Gamma_{\bar{g}}[\langle h_{\mu\upsilon}\rangle = 0]$ 的计算是在极限 $\langle h_{\mu\upsilon}\rangle = 0$ 下完成的，因此在领头阶可以基本忽略 $h_{\mu\upsilon}$ 的变换 (尽管在下一领头阶，以及一般情况下需要包含外 $h_{\mu\upsilon}$ 线的高阶顶点时，这些变换十分重要)。可以直接看出 $\delta_\xi h_{\mu\upsilon}|_{h=0} = \delta^b_\xi \bar{g}_{\mu\upsilon}$ ，这也从另一角度说明，当分解对称性没有反常时，单场有效作用量 $\overline{\Gamma}[\bar{g}_{\mu\upsilon}]$ 应当满足微分同胚不变性。

Appendix B: Covariant Computations and the Heat Kernel

附录 B: 协变计算与热核

The heat kernel method allows to perform covariant computations of effective actions and Feynman-like diagrams in curved space. Here ”covariance” refers to both general covariance, as in General Relativity, and gauge covariance under eventual internal symmetry groups. The heat kernel method is a natural approach to actual computations which are set up by the background field method discussed in the previous appendix.

热核方法可用于在弯曲空间中对有效作用量和类费曼图进行协变计算。此处的“协变”同时指广义相对论中的广义协变性，以及可能存在的内部对称群下的规范协变性。对于前一个附录讨论的背景场方法所确立的实际计算问题，热核方法是一种自然的处理方案。

Rather generally, the application of the background field method on an action $S[\phi]$ results in a Taylor-like expansion $S[\phi] = S[\varphi] + S_1[\varphi,\chi] + S_2[\varphi,\chi] + \cdots$. In many cases of interest, the quadratic part can be written as $S_2[\varphi,\chi] = \frac{1}{2}\chi\cdot\mathcal{O}\cdot\chi$, where we defined an operator of Laplace-type acting on the field’s bundle:

一般而言，对作用量 $S[\phi]$ 应用背景场方法会得到类泰勒展开 $S[\phi] = S[\varphi] + S_1[\varphi,\chi] + S_2[\varphi,\chi] + \cdots$ 。在许多感兴趣的情形中，二次项可以写为 $S_2[\varphi,\chi] = \frac{1}{2}\chi\cdot\mathcal{O}\cdot\chi$ ，其中我们定义了一个作用在场丛上的拉普拉斯型算符:

$$\mathcal{O} = -g^{\mu\upsilon}\nabla_\mu\nabla_\upsilon + E, \tag{122}$$

where $E = E(x)$ is a local endomorphism. In most other cases, kernels which differ from (122) can also be cast in the above form (e.g., the square of a Dirac operator in curved space, $\nabla\!\!\!/$, is $\nabla\!\!\!/^2 = \nabla^2 + \frac{R}{4}$). Everything depends on background fields from now on.

其中 $E = E(x)$ 是一个局部自同态。在大多数其他情形中，与 (122) 形式不同的核也可以整理为上述形式 (例如弯曲空间中狄拉克算符的平方 $\forall$ 即为 $\forall^2 = \nabla^2 + \frac{R}{4}$)。从现在起，所有量都依赖于背景场。

The Green function $G(x, x')$ of (122) is formally defined as the inverse of $\mathcal{O}$ and is the basic building block of covariant Feynman diagrams. It solves the equation:

(122) 的格林函数 $G(x, x')$ 形式上定义为 $\mathcal{O}$ 的逆，是协变费曼图的基本构造块。它满足下述方程:

$$\mathcal{O}_x G(x, x') = \delta^{(d)}(x, x'), \tag{123}$$

where the δ -function is a bi-scalar density such that $\int \sqrt{g} d^d x' \delta^{(d)}(x, x') f(x') = f(x)$. Covariant Feynman diagrams are products of Green functions and vertices acting as differential operators on them. Traditional Feynman diagrams can be seen as the momentum space representations of these products in flat space.

其中 δ -函数是一个双子标量密度，满足 $\int \sqrt{g} d^d x' \delta^{(d)}(x, x') f(x') = f(x)$。协变费曼图是格林函数与作为其上微分算子的顶点的乘积。传统费曼图可以看作这些乘积在平直空间中的动量空间表示。

The heat kernel function of (122) is defined as the solution of the diffusion equation:

(122) 的热核函数被定义为下述扩散方程的解:

$$\partial_s \mathcal{G}(s; x, x') + \mathcal{O}_x \mathcal{G}(s; x, x') = 0, \ \mathcal{G}(0; x, x') = \delta^{(d)}(x, x'). \tag{124}$$

The formal solution is the exponential, $\mathcal{G}(s; x, x') = \langle x' | \mathrm{e}^{-s\mathcal{O}} | x \rangle$, which can be easily related to the Green function by integrating over the diffusion ”time” s :

形式解是指数函数 $\mathcal{G}(s; x, x') = \langle x' | \mathrm{e}^{-s\mathcal{O}} | x \rangle$ ，它可以很容易地通过对扩散“时间” s 积分与格林函数关联起来:

$$G(x, x') = \int_0^\infty \mathrm{d}s \mathcal{G}(s; x, x'). \tag{125}$$

Given that there is a practical way to compute $\mathcal{G}(s; x, x')$, given below, Eq. (125) should be taken as the operative definition of $G(x, x')$.

由于下文会给出计算 $\mathcal{G}(s; x, x')$ 的实用方法，式 (125) 可被当作 $G(x, x')$ 的操作定义。

The heat kernel has an asymptotic expansion for small values of the diffusion ”time” s , known as the Seeley-de Witt expansion, which can be computed iteratively. To justify its form, consider first the solution of (124) in flat space and for $E = 0$

热核在扩散“时间” s 取小值时存在渐近展开，称为西利-德维特展开，可以通过迭代计算得到。为说明该展开的形式，我们首先考虑平直空间中 $E = 0$ 对应下 (124) 的解

$$\mathcal{G}(s;x,x') = \frac{1}{(4\pi s)^{d/2}} e^{-\frac{|x-x'|^2}{4s}}. \tag{126}$$

The flat space formula is generally ”covariantized” as

平直空间的公式通常被“协变化”为

$$\mathcal{G}(s;x,x') = \frac{\Delta(x,x')^{1/2}}{(4\pi s)^{d/2}} e^{-\frac{\sigma(x,x')}{2s}} \sum_{k\geq 0} a_k(x,x')\, s^k. \tag{127}$$

where the Synge-de Witt world function, denoted $\sigma(x,x')$, is half of the square of the geodesic distance between x and x' , and $\Delta(x,x') = (g(x)\,g(x'))^{-1/2} \det\left(-\partial_\mu \partial_{\nu'} \sigma\right)$ is the van Vleck determinant (it is a conventional normalization). In flat space, $2\sigma(x,x') = |x-x'|^2$ and $\Delta = 1$. The bitensors $a_k(x,x')$ are the coefficients of the asymptotic expansion and contain the geometrical information of the operator $\mathcal{O}$ in terms of curvatures of the connection and interactions such as E .

其中，由 $\sigma(x,x')$ 表示的辛奇-德威特世界函数是 x 与 x' 之间测地线距离平方的一半，$\Delta(x,x') = (g(x)\,g(x'))^{-1/2} \det\left(-\partial_\mu \partial_{\nu'} \sigma\right)$ 是范弗莱克行列式 (这是常规归一化)。在平直空间中，$2\sigma(x,x') = |x-x'|^2$ 且 $\Delta = 1$ 。双张量 $a_k(x,x')$ 是渐近展开的系数，包含了算符 $\mathcal{O}$ 关于联络曲率以及 E 这类相互作用的几何信息。

Ultraviolet properties are local in meaningful and renormalizable quantum field theories. In terms of the Green function, locality corresponds to $x \sim x'$, formalized by the coincidence limit $x \to x'$. Given a bitensor $B(x,x')$, the coincidence limit is denoted $[B] = \lim_{x'\to x} B(x,x')$ and regarded as a function of x .

在有意义且可重整化的量子场论中，紫外性质是定域的。对格林函数而言，定域性对应 $x \sim x'$ ，由重合极限 $x \to x'$ 形式化。给定一个双张量 $B(x,x')$ ，其重合极限记为 $[B] = \lim_{x'\to x} B(x,x')$ ，被视为 x 的函数。

The coincidence limits of (the derivatives of) the bitensors $\sigma(x,x')$, $\Delta(x,x')$ and $a_k(x,x')$ can be computed algorithmically starting from few ”crucial” equations. The first two are $\partial_\mu \sigma \partial^\mu \sigma = 2\sigma$ and $\Delta^{1/2} \nabla^2 \sigma + 2\sigma^\mu \partial_\mu \Delta^{1/2} = d\Delta^{1/2}$, while the equations for a_k can be derived by inserting (127) in (124) and expanding in powers of s . The boundary conditions are $[\sigma] = 0, \left[\Delta^{1/2}\right] = 1$, and $a_0(x,x') = 1$ (the identity in the internal space). The coincidence limits relevant for (122) can be arranged by counting the number of covariant derivatives and curvatures, owing to the fact that s is a dimensionful parameter. They are known to a rather high order.

双张量 $\sigma(x,x')$, $\Delta(x,x')$ 和 $a_k(x,x')$ (及其导数) 的重合极限可以从少数几个“核心”方程出发通过算法计算得到。前两个方程为 $\partial_\mu \sigma \partial^\mu \sigma = 2\sigma$ 和 $\Delta^{1/2} \nabla^2 \sigma + 2\sigma^\mu \partial_\mu \Delta^{1/2} = d\Delta^{1/2}$ ，而 a_k 满足的方程可以通过将 (127) 代入 (124) 并按 s 的幂次展开得到。边界条件为 $[\sigma] = 0, \left[\Delta^{1/2}\right] = 1$ ，以及 $a_0(x,x') = 1$ (内空间的单位元)。由于 s 是量纲参数，对应 (122) 的重合极限可以通过计数协变导数和曲率的数量整理得到，目前已知这类量到相当高的阶数。

Using the heat kernel, it is often straightforward to compute the leading one-loop quantum corrections to the effective actions, which are expressed in terms of the trace-log formula:

利用热核，通常可以很方便地计算有效作用量的领头阶单圈量子修正，这些修正可以用迹对数公式表示为:

$$\frac{1}{2}\operatorname{Tr}\log\mathcal{O} = -\frac{1}{2}\int d^d x\sqrt{g}\int\frac{ds}{s}\operatorname{tr}\mathcal{G}(s;x,x)$$
$$\overset{\text{IR reg}}{\to} -\frac{1}{2}\sum_{k\geq 0}\frac{1}{(4\pi)^{d/2}}\int d^d x\sqrt{g}\int\frac{ds}{s^{d/2-1+k}}e^{-sm^2}\operatorname{tr}[a_k]\,, \tag{128}$$

which is true modulo field-independent constants. It is clear that (128) picks up only coincidence limits of the expansion of $\mathcal{G}(s;x,x')$, that is, the coefficients $[a_k]$ upon using the Seeley-de Witt expansion. Each integral over s is generally doubly divergent for massless theories: UV divergences appear for $s\to 0$ and must be subtracted by renormalization, while IR divergences appear for $s\to\infty$ and can be regulated by adding a small mass as e^{-sm^2} inside (128). In dimensional regularization, the limit $m^2\to 0$ can be obtained by opportunely continuing d in each term. In the simplest applications, minimal subtraction selects only one coincidence limit for a given "critical" d . For example, when expanding around the massless theories, $[a_1]$ is the coefficient of the dimensional pole in two-dimensional theories, while $[a_2]$ is the coefficient for four-dimensional ones (modulo normalizations).

该式在忽略与场无关的常数时成立。显然式 (128) 仅提取 $\mathcal{G}(s;x,x')$ 展开的重合极限, 即使用西利-德维特展开后得到的系数 $[a_k]$ 。对 s 的每个积分在无质量理论中通常是双重发散的: $s\to 0$ 会出现紫外发散，必须通过重整化扣除，而 $s\to\infty$ 会出现红外发散，可以通过在式 (128) 中引入小质量 e^{-sm^2} 来规一化。在维数正规化中，可以通过对 d 各项做解析延拓得到极限 $m^2\to 0$ 。在最简单的应用中，最小减除方案仅会给给定的“临界” d 选出一个重合极限。例如，在对无质量理论做展开时，$[a_1]$ 是二维理论中维数极点的系数，而 $[a_2]$ 是四维理论中该极点的系数，忽略归一化不影响结论。

For subleading computations, the ability to handle products of Green functions is paramount. The expansion (127) induces naturally an expansion of the Green function in (125)

对于次领头阶计算，处理格林函数乘积的能力至关重要。展开式 (127) 自然诱导出式 (125) 中格林函数的展开

$$G(x,x') = \sum_{k\geq 0} G_k(x,x')\,a_k(x,x')\,. \tag{129}$$

The leading $G_0(x,x')$ and the subleading $G_k(x,x')$ for $k\geq 1$ are bilocal contributions to the Green function and are determined by a simple integration over the heat kernel parameter s . For example, assuming that d is arbitrary (i.e., not even)

领头阶 $G_0(x,x')$ 和次领头阶 $G_k(x,x')$ 对 $k\geq 1$ 而言是格林函数的双局域贡献，可通过对热核参数 s 做简单积分得到。例如，假设 d 是任意的 (即不一定是偶)

$$G_k(x,x') = \frac{2^{d-2-2k}}{(4\pi)^{d/2}}\frac{\Delta^{1/2}}{(2\sigma)^{d/2-1-k}}\Gamma\left(\frac{d}{2}-1-k\right). \tag{130}$$

If d is even, one must account for logarithmic contributions of the form $\sigma^n\log\sigma$ modifying some order in the k expansion above (as seen in the Hadamard representation of G [24]). When $d\sim 2$ there is an IR ε -pole to subtract in the leading part:

若 d 是偶函数，则必须考虑形式为 $\sigma^n \log \sigma$ 的对数贡献，它会改变上述 k 展开中某一阶的结果 (可见参考文献 [24] 中 G 的阿达马表示)。当 $d \sim 2$ 时，领头阶部分存在一个需要扣除的红外 ε 极点:

$$G_0(x, x')\big|_{d=2-\varepsilon} = \frac{\Delta^{1/2}}{(4\pi)^{d/2}} \frac{\Gamma(d/2-1)}{(2\sigma)^{d/2-1}} + \mu^{-\varepsilon} \frac{\Delta^{1/2}}{2\pi\varepsilon}. \tag{131}$$

If $d \sim 4$, the logarithm is in the subleading part:

若 $d \sim 4$ ，则对数项出现在次领头阶部分:

$$G_1(x, x')\big|_{d=4-\varepsilon} = \frac{\Delta^{1/2}}{(4\pi)^{d/2}} \frac{\Gamma(d/2-2)}{(2\sigma)^{d/2-2}} + \mu^{-\varepsilon} \frac{\Delta^{1/2}}{8\pi^2\varepsilon}. \tag{132}$$

The scale μ is the reference scale of dimensional regularization. The subtractions are the covariant analog of the infrared poles appearing in dimensionally regulated momentum-space Feynman diagrams.

标度 μ 是维数正规化的参考标度。此处的减除是维数正规化动量空间费曼图中出现的红外极点的协变对应。

Covariant Feynman Diagrams

协变费曼图

Leading quantum corrections at one loop are generally covered by the simple use of (128) for as long as the kernel of the quadratic part of the bare action can be written in terms of a Laplace-type operator using the background method. However, computations of subleading corrections (or corrections to composite operators) generally involve more complicate structures that can be diagrammatically summarized as covariant Feynman diagrams, which can be seen as the coordinate representation of the standard diagrams in momentum space [52]. In this appendix, we follow the presentation of Appendix C of Ref. [62] that summarizes some of the most important points of Ref. [52].

只要裸作用量二次项的核可以利用背景法写成拉普拉斯型算子的形式，一阶领头量子修正通常就可以通过简单套用 (128) 处理。然而次领头修正 (或复合算子修正) 的计算通常涉及更复杂的结构，这些结构可以用图总结为协变费曼图，可看作动量空间标准图的坐标表示 [52]。本附录我们遵循文献 [62] 附录 C 的表述，该表述总结了文献 [52] 部分最重要的核心内容。

In practical applications, covariant Feynman diagrams are constructed as products of Green functions, and the vertices act as multilocal differential operators on them. Using the case $d \sim 4$ as example, it is convenient to rewrite (129) as

在实际应用中，协变费曼图由格林函数乘积构造得到，顶点对格林函数充当多局域微分算子。我们以 $d \sim 4$ 情形为例，可方便地将 (129) 改写为

$$G(x, x') = G_0(x, x') + G_1(x, x')\, a_1(x, x') + \bar{G}(x, x'), \tag{133}$$

which isolates the first two contributions as potential sources of divergences in the limit $x \sim x'$ from the rest, denoted $\bar{G}(x, x')$, which is regular in the limit. Beyond one loop, diagrams will contain several structures: local poles of the form $\frac{1}{\varepsilon}$ that contribute to the renormalization group through beta functions, higher-order poles up to the form $\frac{1}{\varepsilon^L}$ for $L \geq 2$ at the L-th loop order, and divergences multiplying the regular nonlocal part $\bar{G}(x, x')$, besides finite nonlocal structures that depend on $\bar{G}(x, x')$. In meaningful renormalizable field theories, the local divergences must be subtracted, but only the leading $\frac{1}{\varepsilon}$ one contributes to the renormalization group running. For example, two-loop divergences can only appear in the combination $\frac{\mu^{-2\varepsilon}}{\varepsilon^2} - 2\frac{\mu^{-\varepsilon}}{\varepsilon^2}$, where the first term is a genuine two-loop divergence, while the second comes from a one-loop divergence multiplying the one-loop counterterm. The logarithmic derivative with respect to μ is insensitive to this structure; in fact the divergences cancel in the limit $\varepsilon \to 0$. The divergences multiplying nonlocal structures of $\bar{G}(x, x')$ must cancel, or else the theory is nonlocal in the ultraviolet (i.e., nonrenormalizable), which is often an important and nontrivial check of the regularization process.

该式将前两项贡献分离出来，作为 $x \sim x'$ 极限下的潜在发散源，其余部分记为 $\bar{G}(x, x')$，在该极限下是正则的。一阶圈以上，费曼图会包含多种结构: 形式为 $\frac{1}{\varepsilon}$ 的局域极点，它通过 β 函数对重整化群有贡献；对于 L 阶圈图的 $L \geq 2$，最高极点可达 $\frac{1}{\varepsilon^L}$ 形式；还有乘在正则非局域部分 $\bar{G}(x, x')$ 上的发散，此外还有依赖 $\bar{G}(x, x')$ 的有限非局域结构。在有意义的可重整场论中，局域发散必须被减除，但只有领头的 $\frac{1}{\varepsilon}$ 发散对重整化群跑动有贡献。例如，两圈发散只能出现在组合 $\frac{\mu^{-2\varepsilon}}{\varepsilon^2} - 2\frac{\mu^{-\varepsilon}}{\varepsilon^2}$ 中，其中第一项是真正的两圈发散，第二项来自一阶圈发散乘一阶圈抵消项。对 μ 的对数导数不敏感于该结构；事实上发散会在 $\varepsilon \to 0$ 极限下抵消。乘在 $\bar{G}(x, x')$ 非局域结构上的发散必须抵消，否则该理论在紫外是非局域的 (即不可重整的)，这通常是正则化过程中一项重要且非平凡的检验。

Using (133), we can see that building blocks of the Feynman diagrams are structures of the form $Q(x, x')\,\sigma(x, x')^{-b}$, where $Q(x, x')$ depends on the vertices of the theory and b on the number of propagators. The covariant generalization of dimensional regularization starts with the basic relation:

利用 (133) 可以看出，费曼图的构造块是形式为 $Q(x, x')\,\sigma(x, x')^{-b}$ 的结构，其中 $Q(x, x')$ 依赖理论的顶点，b 依赖传播子的数量。维正规化的协变推广从基本关系出发:

$$\frac{1}{\sigma(x, x')^{\frac{d}{2} - c\varepsilon}} \sim \frac{(2\pi)^{\frac{d}{2}}}{c\varepsilon\Gamma(d/2)} \mu^{-2c\varepsilon} \delta^{(d)}(x, x'), \tag{134}$$

which establishes equivalence for $x \sim x'$ and the reference scale μ to preserve the dimensionality (this relation can be proven using Riemann normal coordinates in curved space). In (134), the dimension d is the analytically continued dimension.

该式建立了 $x \sim x'$ 和参考标度 μ 之间的等价关系以保持量纲 (该关系可利用弯曲空间中的黎曼法坐标证明)。(134) 中的量纲 d 是解析延拓后的维数。

Higher inverse powers of the world function are also divergent. They can be obtained using

世界函数的更高次逆幂也发散，可利用下式得到:

$$\left(\nabla^2 - Y\right) \frac{\Delta^{1/2}}{\sigma^b} = b\left(2(b+1) - d\right) \frac{\Delta^{1/2}}{\sigma^{b+1}}, \; Y(x, x') \equiv \Delta^{-1/2} \nabla^2 \Delta^{1/2}, \tag{135}$$

which can be proven easily applying the crucial relations. For example,

应用核心关系可以很容易证明该式，例如:

$$\frac{\Delta^{1/2}}{\sigma(x,x')^{\frac{d}{2}+1-c\varepsilon}} \sim \frac{(2\pi)^{\frac{d}{2}}\mu^{-2c\varepsilon}}{c\varepsilon d\Gamma(d/2)}\left(\nabla^2-\frac{R}{6}\right)\delta^{(d)}(x,x'), \tag{136}$$

which is obtained inverting (135) for $b = d/2$ and using the coincidence limits of the biscalars $\left[\Delta^{1/2}\right] = 1$ and $[Y] = R/6$.

它是通过对 $b = d/2$ 反解 (135)，并利用双标量 $\left[\Delta^{1/2}\right] = 1$ 和 $[Y] = R/6$ 的重合极限得到的。

Whenever a bilocal tensor $Q(x,x')$ "touches" the Dirac delta on the righthand side of (136), it can be replaced with its coincidence limit in the divergent part $Q(x,x')\delta^{(d)}(x,x') = [Q]\delta^{(d)}(x,x')$. If bilocal operators are separated from the Dirac delta by covariant derivatives, it is necessary to integrate by parts. For example, we could manipulate as follows:

每当双局部张量 $Q(x,x')$ 触及 (136) 右侧的狄拉克 δ 函数时，在发散部分 $Q(x,x')\delta^{(d)}(x,x') = [Q]\delta^{(d)}(x,x')$ 中都可以将其替换为它的重合极限。如果双局部算子通过协变导数与狄拉克 δ 函数隔开，则需要进行分部积分。例如，我们可以按如下方式变换:

$$Q(x,x')\nabla_\mu\delta^{(d)}(x,x') = \nabla_\mu\left(Q(x,x')\delta^{(d)}(x,x')\right) - \nabla_\mu Q(x,x')\delta^{(d)}(x,x')$$

$$\sim \nabla_\mu\left([Q]\delta^{(d)}(x,x')\right) - \left[\nabla_\mu Q\right]\delta^{(d)}(x,x').$$

(137)

Similar manipulations can be performed for the cases with more derivatives. Detailed applications of the heat kernel method at leading and nonleading orders can be found in [52].

对于含更多导数的情况也可以进行类似变换。热核方法在领头阶和非领头阶的详细应用可见文献 [52]。

Cross-References

交叉引用

- Asymptotic Safety of Gravity with Matter

- 含物质的引力渐近安全

- One-Loop Divergences in Higher-Derivative Gravity

- 高阶导数引力中的一圈发散

- Quantum General Relativity and Effective Field Theory

    - 量子广义相对论与有效场论

- The Background Information About Perturbative Quantum Gravity

    - 微扰量子引力背景知识

- The Functional Renormalization Group in Quantum Gravity

    - 量子引力中的泛函重整化群

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